ENTROPY IN PORTFOLIO OPTIMIZATION

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ABSTRACT

In this thesis, we investigate the properties of entropy as an alternative measure of risk. Entropy has been compared with the traditional risk measure, variance from different point of views. It has been established that though variance is computationally simple and very popular among practitioners, a more flexible measure of risk is demanded to cope with the uncertainty in real data that are typically non-normally distributed. Entropy, however, is not computationally easy but is not restricted to the assumption of normality. In this study we explore and investigate the application of entropy in portfolio models. More specifically, we use multi-objective models that are the mean-entropy-entropy (MEE). The purpose of this new model is to overcome the limitations as observed in a traditional model; that is, having performance close to Markowitz's mean-variance (MV) model when data comes from a normal distribution, but exhibit better performance when data comes from a non-normal distribution. The special advantage of the new model is that it is more diversified than any other models available in the literature. Also in this thesis, we address the issue of robust estimation of entropy. Special attention has been paid to entropy estimation with kernel density, which is popular among practitioners. The failure of this technique has been investigated and an adaptive beta-divergent method is proposed to ensure robust estimation. The usefulness of this technique has been verified with Monte-Carlo simulation in the context of portfolio analysis. Details of the algorithms which include entropy estimation which would enhance the application of a proper risk measure like entropy, is provided. Finally, the models are compared with Monte-Carlo simulation experiments and real data examples.

ABSTRAK

Dalam tesis ini, kami menyiasat sifat entropi sebagai ukuran risiko alternatif. Entropi dibandingkan dengan ukuran risiko tradisional, varians, dari sudut pandangan yang berbeza. Adalah sedia diketahui, walaupun varians adalah mudah dihitung dan sangat popular dikalangan pengguna, keperluan satu ukuran risiko yang lebih fleksibel adalah diharapkan dalam menghadapi ketidaktentuan dalam data sebenar yang biasanya bertaburan bukan normal. Entropi, bagaimanapun, bukan mudah dihitung tetapi ia tidak terhad kepada andaian taburan normal. Didalam kajian ini saya meneroka dan menyiasat penggunaan entropi dalam model portfolio. Lebih spesifik penggunaan model pelbagai objektif digunakan, iaitu min-entropi-entropi (MEE). Tujuan model baru ini adalah untuk mengatasi keterbatasan sebagaimana yang dilihat dalam model tradisional; iaitu, ia menghampiri min-varians (MV) Markowitz apabila data bertaburan normal, tetapi juga mempamerkan prestasi yang lebih baik apabila data tidak bertaburan normal. Kelebihan utama model ini adalah ianya lebih pelbagai daripada model-model yang sedia adadalam literatur. Tesis ini juga melihat isu anggaran teguh entropi. Perhatian khas dibuat ke atas anggaran entropi dengan kaedah kernel ketumpatan yang popular di kalangan pengguna. Kegagalan kaedah ini telah disiasat dan satu beta-divergent di cadangkan untuk memastikan keteguhan anggaran. Kegunaan teknik ini telah disahkan melalui simulasi Monte-Carlo dalam konteks analisis portfolio. Algorithma yang terperinci bagi anggaran entropi juga diberikan, bagi meningkatkan penggunaan ukuran risiko yang sesuai seperti entropi. Model-model sedia ada dan yang dibangunkan dalam kajian ini, dibandingkan melalui eksperimen simulasi Monte-Carlo dan contoh data sebenar.

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TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

LIST OF SYMBOLS AND ABBREVIATION

AIC	Akaike's Information Criterion
ASR	Adjusted for Skewness Sharpe ratio
BCV	Biased Cross Validation
CDaR	Conditional Drawdown-at-Risk
CE	Cross-Entropy
CVaR	Conditional value at risk
$E[X]$	Mean for random variable X
$E[X^2]$	Second moment for random variable X
$E[X^k]$	k-th moment for random variable X
$f_X(x)$	Probability density function for a random variable X
$f'_X(x)$	First deviation of probability density function for a random variable X
$F'_X(\mathrm{x})$	First deviation of cumulative density function for a random variable X
FTR	Farinelli and Tibiletti ratio
GCE	Generalized Cross-Entropy
H(X)	Entropy
$H(x,y)$	Joint Entropy

$H(y\|x)$	Conditional Entropy
IMSE	Integrated Mean Squared Error
MAD	Mean Absolute Deviation
MADR	Mean Absolute Deviation Ratio
ME	Mean-Entropy
$M_X(x)$	Moment generation function for a random variable X
MI	Mutual Information
MLE	Maximum Likelihood Estimations
MSE	Mean Squared Error
PT	Portfolio Turnover
SCV	Smooth Cross Validation
SR	Sharpe ratio
SSR	Sortino-Satchell ratio
SR_{out}	Average of Out-Of-Sample Estimate of the SR
SR_{in}	Average of In-Sample Estimate of the SR
VaR	$\text{Value} - \text{at} - \text{Risk}$
$Var(X)$	Variance for a random variable X

LIST OF APPENDICES

CHAPTER 1: INTRODUCTION

1.1 General introduction

Entropy like variance is a collective measure of uncertainty but unlike variance, it can be applied on both cardinal and ordinal variables. Entropy is concerned with probabilities as a measure of disorder. It represents the investor's average uncertainty of the returns of a project, and being distribution free, it is not affected by errors due to the fitting of the distribution of returns to a particular distribution. McCauley (2003) argues that entropy has the ability to capture the complexity of the systems without requiring rigid assumptions that can bias the results obtained. Interest in relating entropy to variance dates back to Shannon (1948) who proposed comparison of continuous random variables according to the entropy power fraction defined as the variance of a Gaussian random variable with given entropy. The performance and feasibility of entropy as a measure of uncertainty are compared with variance in several studies that established entropy as an alternative measure of dispersion (Maasoumi, 1993; Soofi, 1997). According to Ebrahimi et al. (1999), these two measures use different metrics for concentration. Unlike the variance which measures concentration only around the mean, the entropy measures diffuseness of the density irrespective of the location. They examine the role of variance and entropy in ordering distributions and random prospects, and conclude that there is no universal relationship between these measures in terms of ordering distributions. These authors found that, under certain conditions, the order of the variance and entropy is similar when continuous variables are transformed also using a Legendre series expansion shows that entropy may be related to high-order moments of a distribution which, unlike the variance, could offer a much closer characterization of probability since it uses much more information about the distribution than the variance. Noting the same point, Maasoumi and Racine (2002)

argue that in the case that the empirical probability distribution is not perfectly known, the entropy constitutes an alternative measure for the uncertainty, predictability and goodness-of-fit.

Unlike variance, estimation of entropy from real data is not straightforward. Once the density function is known, the entropy can be estimated using plug-in or resubstitute estimator (see Cover and Tomas, 1991; Beirlant, 1997). However seldom do we know the true density for the available data. Dmitriev and Tarasenko (1973) and Ahmad and Lin (1976) address the plug-in estimate of entropy using kernel density estimator. This established estimator is consistent but bias increases with the dimension of data. The resubstitute estimate of entropy with kernel density also provide consistent estimator for dimension that are not more than 3 (see Hall and Morton, 1996 and Ivanov, 1981). The consistency of histogram based entropy estimation is established by Györfi and van der Meulen (1987) and Hall and Morton (1993). Applications of this estimator in real data are found in Moddemeijer (1999) and Darbellay and Vajda (1999).Vasicek (1976) proposed sample spacing estimator for entropy estimation from real data. A modified version of this estimator is offered by Correa (1995). Consistency and asymptotic properties are studied in (Tsybakov et al., 1996) and Beirlantand Zuijlen (1985). The nearest neighborhood estimator of entropy is proposed by Kozachenko and Leonenko (1987) and its consistency properly verified by Tsybakov and van der Meulen (1994). In a recent work, Gupta and Srivastava (2010) introduce Bayesian parameter estimation for entropy.

Portfolio optimization has been the object of intense research and is still developing. Markowitz's (Markowitz, 1952) mean-variance (MV) efficient portfolio selection is one of the most widely used approaches in solving portfolio diversification problem and is very popular among practitioners. However, some drawbacks of this approach are

pointed out in the literature. Bera and Park (2008) argue that MV approach, based on sample moments like mean and variance, often concentrates on a few assets only and this leads to less diversified portfolio. Due to less attention to uncertainty in the data and adoption of a wrong model, sample estimates of mean and variance can be poorly estimated (Jobson and Korkie, 1980) and hence portfolio optimization based on inaccurate point estimates may be highly misleading. Sometimes, variations in the input data may affect the portfolio greatly and even a few new observations may change the portfolio completely. In addition, empirical evidences show that almost all asset classes and portfolios have returns that are not normally distributed (Xiong et al., 2011), and the first and second moments are generally insufficient to explain portfolios in the case of non-normal return distribution (Usta and Yeliz, 2011). Ke and Zhang (2008) notify another limitation of MV model that the standard deviation cannot perfectly represent the risk, because the sign of error does not affect the fluctuation. However, many assets' return distributions are asymmetrical. In addition, most asset return distributions are more leptokurtic, or fatter tailed, than are normal distribution. Patton (2004) showed that knowledge of both skewness and asymmetric dependence leads to economically significant gains. Recent research (Müller, 2010, for example) suggests that higher moments are important considerations in asset allocation. Investors are particularly concerned about significant losses that are the downside risk, which is a function of skewness and kurtosis. There are few studies with conclusion that the out-of-sample performances of the MV portfolios are not quite sufficient (Bear and Park, 2008andJordon, 1985).

Through the works of Philippatos and Gressis (1975), Kapur and Kesavan (1992), Samanta and Roy(2005), Hoskisson et al. (2006), Jana et al. (2007) and Jana et al. (2009), it is now established that in order to measure the diversification, entropy is a widely accepted measure. Philippatos and Wilson (1972) introduce entropy in finance as

a tool for portfolio optimization. Their comparative analysis between the behaviors of the standard deviation and the entropy in finance conclude that entropy is more general and has some advantages over standard deviation. In another study, Saxena (1983) used entropy to select the best alternative investment projects. Nawrocki and Hardling (1986) verify investment performance when entropy is used as a measure of risk. He suggested a heuristic algorithm using portfolio analysis with state-value weighting entropy as a measure of investment risk. Philippatos and Gressis (1975) provide conditions in which mean-variance, mean-entropy and second degree stochastic dominance are equivalent.

It is well known that the sample mean vector and covariance matrix, basic elements of portfolio analysis, are sensitive to outlying observations. A little amount of contamination may have huge effect on their estimate and a dramatic change may occur on the output of portfolio analysis (Demiguel and Nogales, 2009). Being non-parametric, entropy based portfolio model has its own merit. However, entropy itself may be poorly estimated in the presence of contamination (Escolano et al., 2009) and, thus, asset allocation based on it could be misleading in some situations. Therefore, like other procedures, the robustness of entropy estimation should also be verified.

1.2 Literature Review

Entropy and information theory analysis became very popular in the finance and economics literature during the early 1970.Anumber of articles demonstrate that entropy analysis measures meaningful information that is not available to standard statistical techniques such as variance or correlation analysis. Though Horowitz (1976) claims that there should not be any statistical measure like entropy that tells whether information is meaningful in an economic sense, Philippatoes and Wilson (1972, 1975) defend that being nonparametric entropy is a better statistical measure of risk than the variance. Wyner and Ziv (1969) provided a bound on entropy in terms of a single moment of a

continuous random variable. This entropy-moment inequality, for which the variance is a special case, has played an important role in the development of prediction theory (Shepp et al., 1979). Maasoumi and Theil (1979) gave approximations for two entropy-based income disparity measures in terms of the first four moments of the underlying income distributions. Chandra and Singpurwalla (1981) discussed entropy ordering in the context of some notions common between economics and reliability analysis. Mukherjee and Ratnaparkhi (1986) presented some relationships between the entropy and variance for a number of distributions, graphically. Smaldino (2013) exhibit two common measures of the uncertainty inherent in a distribution of possible outcomes are variance and entropy, yet there is currently no standard measure. For small numbers of discrete possible outcomes, Smaldino noted that variance is the better measure because it captures the spread between outcomes as well as their differential possibilities. However, variance can categorically fail as a measure of uncertainty when distributions are multimodal or discontinuous, in which case entropy should be used to characterize uncertainty.

Popkov, (2005) proposed entropy-optimal investment portfolio which allows one to take into consideration the investor's response to the reachable income. The author focus on the computational methods adapted to the problems arising in these models. Huang (2008) proposes two types of credibility-based fuzzy mean-entropy models. Entropy is used as the measure of risk, the smaller the entropy value is, the less uncertainty the portfolio return contains, and thus, the safer the portfolio. Furthermore, as a measure of risk, entropy is free from reliance on symmetrical distributions of security returns and can be computed from nonmetric data. In addition, it compares the fuzzy mean-variance model with the fuzzy mean-entropy model in two special cases and presents a hybrid intelligent algorithm for solving the proposed models in general cases. Wand and Pan (2010) applied entropy as a measure of risk in air defense

disposition problem which is full of uncertainties and risks in modern war, the smaller entropy value is, the safer the dispositions. Within the frame work of uncertainty theory, two types of fuzzy mean-entropy models are proposed, and a hybrid intelligent algorithm is presented for solving the proposed models in general cases.

Ke and Zhang (2008) integrate the entropy theory into Markowitz portfolio model to make a better performance in simulation for the relation between investment return and risk. They argue that this model provides a natural probabilistic interpretation for daily return which usually changes from positive to negative, and it indicates that the entropy can be used as a complement to the mean-variance portfolio model. Bera and Park (2008) provide an alternative for portfolio selection model by introducing cross-entropy (CE) and generalized CE (GCE) as the objective functions. This automatically captures the degree of imprecision of the mean and covariance matrix estimates. Usta and Yeliz (2010) added the entropy theory to the mean-variance-skewness model (MVSM) to generate a well-diversified portfolio. They present a multi-objective model which includes mean, variance and skewness of the portfolio as well as the entropy of portfolio weights and compare its performance with traditional models in terms of a variety of portfolio performance measures. Their finding is that smaller portfolio turnover is achieved when all the variance, skewness and entropy are included in the objective function. We can hardly find such studies that evaluate if entropy based portfolio model alone can capture the asymmetry in the assets. This verification is necessary because if entropy itself can capture the asymmetry, adding skewness in the objective function is redundant. Although the superiority of entropy is highlighted in a number of papers, it is still not popular among practitioners since unlike MV the ready-to-use computational detail for entropy based portfolio is not easily available. Bhattacharyya et al. (2009) proposed fuzzy mean-entropy-skewness models for optimal portfolio selection. Entropy is favored as a measure of risk as it is free from dependence on symmetric probability

distribution. Yu and Lee (2011) compared five portfolios rebalancing models, with consideration of transaction cost and consisting of some or all criteria, including risk, return short selling, skewness, and kurtosis to determine the important design criteria for a portfolio model. They argue that rebalancing models which consider transaction cost, including short selling cost, are more flexible and their results can reflect real transactions. Yu et al. (2014) compare the mean-variance efficiency, realized portfolio values, and diversity of the models incorporating different entropy measures by applying multiple criteria method and conclude that including entropy in models enhances diversity of the portfolios and makes asset allocation more feasible than the models without incorporating entropy. Bhattacharyya et al., (2014) proposed fuzzy stock portfolio selection models that maximize mean and skewness as well as minimize portfolio variance and cross-entropy. To quantify the level of discrimination in a return for a given value of return, cross-entropy is used. To capture the uncertainty of stock returns, triangular fuzzy numbers are considered. The authors claim that their proposed model has better empirical performance than the others.

In recent literature, more attention has been paid on the robust estimation of return and risk and on the robust optimization of portfolio analysis as well. Schied (2006) give a survey on recent developments in the theory of risk measures. He discusses risk measures and associated robust optimization problems in the frame work of dynamic financial market models. Lobo and Boyd (2000), Costa and Paiva (2002), Halldorsson and Tutuncu (2003) and Lu (2006) address the robust mean-variance portfolio considering uncertainties in the parameters involved in the mean and the covariance matrix and recommend using interior-point algorithms. The uncertainty is further addressed in the work of Zymler et al., (2011). The robust linear programming approach has been introduced by Ben-Tal et al., (2000) to formulate a robust multistage portfolio analysis. El Ghaoui et al., (2003) investigated the robust portfolio optimization using

worst-case VaR, where only partial information on the distribution is known. Goldfarb and Lyengar (2003) also consider the robust VaR portfolio selection problem by assuming a normal distribution. Ferties (2012) pay special attention to the robustness of risk measures where a robust version of CVaR and an entropy based risk measure are introduced. Glasserman and Xu (2014) develop a frame work for quantifying the impact of model error and for measuring and minimizing risk in a way that is robust to model error. Using relative entropy to constrain model distance leads to an explicit characterization of worst-case model errors; this characterization lends itself to Monte-Carlo simulation, allowing straight forward calculation of bounds on model error with very little computational effort beyond that required to evaluate performance under the baseline nominal model. This approach goes well beyond the effect of errors in parameter estimates to consider errors in the underlying stochastic assumptions of the model and to characterize the greatest vulnerabilities error in a model. Recently, a data driven portfolio optimization technique has been proposed by Calafiore (2013). Lagus et al. (2015) use coherent and distortion risk measure in their robust portfolio optimization.

Evaluation of the out-of-sample performance and diversification of the traditional model MV and its extensions suggest that there are still many avenues for improvements, needed in order to gain a better diversified portfolio model with higher out-of-sample performance. These will be the main emphasis of the study.

1.3 Motivation and Objectives

The information theoretic construction of entropy has been used in a variety of fields since its introduction in 1948 by Claude Shannon. This concept of entropy, in an analogy to the identically named object in statistical physics, is concerned with uncertainty of the outcome of a random variable. In recent years entropy has been applied to problem beyond those in communication theory, for which it was initially

developed, infields as varied as image processing, physics, economics, biology, and, as is the concern of this work, financial modeling.

Uncertainty is a very common phenomenon of financial market and the only satisfactory description of uncertainty is the probability. Therefore, any measure the uncertainty should be in the form of probability. From this point of view, entropy is a more general measure than variance since entropy is a function of a probability distribution. Although the MV model is the pioneer of portfolio analysis, current practitioners are looking for some variant of this model to characterize the real data features. In searching for better discretion of reality, academics are involved in developing complex model (for example, in corporation of fuzzy logic in portfolio) that are sometimes computationally expensive or difficult to interpret. In this context, entropy based portfolio model can be a better alternative since entropy can provide risk measure as well as capturing uncertainty adequately; it is non-parametric and it is not restricted to normality assumption; by definition it is measure of diversity. Apart from verifying entropy as an alternative measure of risk and evaluate if entropy based portfolio model can overcome the limitations of Markowitz portfolio.

The main objective of this study is to establish an alternative model, Mean-Entropy-Entropy, which aims at optimizing a portfolio with less risk and more diversified than traditional models. This is done through:

1. verifying working capability of entropy based portfolio models in real data
2. study limitations and remedies of entropy based portfolio optimization
3. provide robust procedure of risk measurement and portfolio analysis
4. provide complete guidelines for portfolio optimization based on entropy

1.4 Outline of the Thesis

The rest of the thesis is organized as follows. Chapter 2 presents detail background of portfolio analysis. The existing models such as Mean-Variance and Mean-Entropy portfolios with their variants and different risk measures such as variance, semi variance, MAD, VaR, CVaR and Entropy are discussed with their application procedure and shortcomings.

Chapter 3 discusses entropy estimation in detail. For this, we discuss different issues of density estimation such as number of bin selection for histogram and bandwidth selection for kernel density. We discuss the technical detail of entropy computation with R. We also compare estimation of entropy from histogram and kernel density. A comparison is made between entropy and variance to ascertain which of these provides a much meaningful characterization as a risk measure.

In Chapter 4, we focus on multi objective portfolio models. We suggest a new nonparametric and well diversified multi objective portfolio model, MEE where both measures risk and diversity, are controlled by entropy. This model is evaluated with real and simulated data and comparison has been made with some benchmark models.

In Chapter 5 the robustness of entropy measure is verified. Since the kernel density is robust up to certain level, a new highly robust method for estimating kernel density and entropy is proposed; this is verified and compared with traditional approach via simulation. The application of the new roust procedure has been discussed in context of portfolio analysis.

Finally, Chapter 6 contains discussions and conclusions.

CHAPTER 2: TRADITIONAL PORTFOLIO MODELS

2.1 Introduction

A portfolio is a collection of investments all owned by the same individual or organization. These investments include securities and financial assets, like stocks, bonds, and mutual funds. Investments of a portfolio are usually diversified among risky and risk free asset. A risky asset is an investment with a return that is not guaranteed and each asset carry varying levels of risk. For example, holding a corporate bond is generally less risky than holding a stock. The risk-free asset is the (hypothetical) asset which pays a risk-free rate. In practice, short-term government securities (such as US treasury bills) are used as a risk-free asset, because they pay a fixed rate of interest and have exceptionally low default risk. The risk-free asset has zero variance in returns (hence is risk-free); it is also uncorrelated with any other asset (by definition, since its variance is zero). Treasury bills are the least risky and the most marketable of all money market instruments. They are considered to have no risk of default, have very short-term maturities, have a known return, and are traded in active markets. They are the closest approximation that exists to a riskless investment.

2.2 Traditional Portfolio Selection Models

In portfolio theory, given a set of assets, the portfolio selection problem is to find the optimum way of investing a particular amount of money in these assets. Each possible strategy is considered as a portfolio selection model. In this section, we present the well-known traditional portfolio selection models and also provide definitions and notations required in this study.

2.2.1 Equally Weighted Model (EW)

Equally weighted (EW) model considers the portfolio weights to be equal $x_i = \frac{1}{n}$ for $i = 1, 2, \dots, n$and does not involve any optimization or estimation, besides; it completely ignores the mean and variance of return. This naive rule for asset allocation has been extensively used by investors although a number of complicated derived models have been developed. Moreover, various studies in the literature such as Bloomfield et al. (1997); Jordon (1985); Bear and Park (2008); DeMiguel (2009) show that the EW portfolio works well for the out-of-sample cases. There are two reasons for using the naive rule as a benchmark. First, it is easy to implement because it does not rely either on estimation of the moments of asset returns or on optimization. Second, despite the sophisticated theoretical models developed in the last 50 years and the advances in methods for estimating the parameters of these models, investors continue to use such simple allocation rules for allocating their wealth across assets.

2.2.2 Optimal Mean-Variance Portfolio

Harry Markowitz (1952, 1959) developed his portfolio-selection technique, called modern portfolio theory (MPT). Prior to Markowitz's work, security-selection models focused primarily on the returns generated by investment opportunities. The standard investment advice was to identify those securities that offered the best opportunities for gain with the least risk and then construct a portfolio from these. The Markowitz theory retained the emphasis on return; but it elevated risk to a coequal level of importance, and the concept of portfolio risk was born. While risk has been considered an important factor with variance as an accepted way of measuring risk, Markowitz was the first to clearly and rigorously show how the variance of a portfolio can be reduced through the impact of diversification. He proposed that investors focus on selecting portfolios based

on their overall risk-reward characteristics instead of merely compiling portfolios from securities that each individually that have attractive risk-reward characteristics.

The main goal of portfolio selection is to obtain optimum weights associated with assets that minimize the risk of the portfolio subject to the portfolio's attaining some target expected rate of return. In other words, a portfolio $x = (x_1, x_2, \ldots, x_n)$ is a vector of weights that represents the investor's relative allocation of the wealth satisfying

$$\sum_{i=1}^{n} x_i = x' 1_n = 1, \quad \text{where } 1_n \text{ is a } n \times 1 \text{ vector of ones.} \tag{1}$$

In Markowitz mean-variance framework (Markowitz, 1952), the sample variance is used as the measure of risk and sample mean as a measure of return. Thus, the mean-variance (MV) problem chooses weights, which minimizes the variance of the portfolio return subject to a pre-determined target, as follows

$$\min_x x' \sum x \tag{2}$$

$$\text{s.t.} \quad E(x'R) = x' \; m = \mu_0 \,, \; x' 1_n = 1$$

where $\Sigma = Var(R)$ and $m = E(R)$ of asset return vector, $R = (R1, R2, \ldots, Rn)$. Alternatively,

$$\max_x x' m \tag{3}$$

$$\text{s.t.} \quad x' \Sigma \, x = d_0, \quad x' 1_n = 1.$$

Mean-variance analysis is based on a single period model of investment. At the beginning of the period, the investor allocates his wealth among various asset classes, assigning a nonnegative weight to each asset. During the period, each asset generates a random rate of return so that at the end of the period, his wealth has been changed by

the weighted average of the returns. In selecting asset weights, the investor faces a set of linear constraints, one of which is that the weights must sum to one.

2.2.2.1 Assumptions and Limitations of MV

As with any model, it is important to understand the assumptions of mean-variance analysis in order to use it effectively. The MV model is based on several assumptions concerning the behaviour of investors and financial markets:

1. A probability distribution of possible returns over some holding period can be estimated by investors.
2. Investors have single-period utility functions in which they maximize utility within the framework of diminishing marginal utility of wealth.
3. Variability about the possible values of return is used by investors to measure risk.
4. Investors care only about the means and variance of the returns of their portfolios over a particular period.
5. Expected return and risk as used by investors are measured by the first two moments of the probability distribution of returns-expected value and variance.
6. Return is desirable; risk is to be avoided.
7. Financial markets are frictionless.

However, in reality, these assumptions may not always be true. One limitation of MV is that it is restricted to the normally distributed assets, which depend on only the first two moments. However, financial returns are typically non-normal (Bates, 1996; Jorion, 1988; Hwang and Satchell, 1999; Harvey and Siddiqui, 1999; 2000; Bonato, 2010; Zuluaga and Cox, 2010; Xiong et al., 2011) and exhibit negative skewness, severe excess kurtosis (Bonato, 2011) and some form of asymmetric dependence (Erb et al., 1994; Longin and Solnik, 2001; Ang and Bekaert, 2002; Ang and Chen, 2002; Campbell et al., 2002; Bae et al., 2003; Patton, 2004). According to Xiong et al. (2011),

investors are concerned about the significant losses which are related to the skewness and kurtosis and a portfolio based on only mean and variance neglects investors' preferences. Recent researches (Müller, 2010, for example) suggest that higher moments are important considerations in asset allocation.

The instability and ambiguity of MV optimization is that it magnifies the impact of estimation errors (Michaund, 1998). Thus, inaccuracy in point estimate of mean and variance may result in highly misleading optimization. Sometimes, variations in the input data may affect the portfolio greatly and even a few new observations may change the portfolio completely. The success of the portfolio thus partially depends on the proper estimate of the risk. However, even if the risk is estimated properly from historical data, the problem of MV portfolio may not be resolved since it does not pay proper attention to the uncertainty of the data (Bera and Park, 2008; Usta and Yeliz, 2010); MV often concentrates only on few assets. Therefore, an MV optimal portfolio may be less diversified and its out-of-sample performance is not as good as the naive 1/N benchmark (Jorion, 1985; DeMiguel, 2009). Ke and Zhang (2008) notify another limitation of MV model that the standard deviation cannot perfectly represent the risk, because the sign of error does not affect the fluctuation.

2.3 ALTERNATIVE TO MV PORTFOLIO

2.3.1 Alternative risk measures

Many studies have proposed alternative risk measures in line with the motivation for overcoming the limitations of variance. At least four alternative risk measures, namely Semi variance (SV), Mean Absolute Deviation (MAD), Value at risk (VaR), Minimax and Entropy are found in real state literature.

2.3.1.1 Semi variance (SV)

Variance as a risk measure for portfolio selection is questioned by many researchers because variance penalizes both returns above and below expected return. But for an investor, risk is any possibility of getting below what he expects. Downside risk measures quantify possibilities of return below expected return. Markowitz (1959) suggested a downside risk measure known as semi variance (SV). Semi variance is the expected value of the squared negative deviations of possible outcomes from the expected return. The definitions derived as follows:

$$SV_{\mu} = E[(R-\mu)^{-}]^2, \tag{4}$$

where $(R-\mu)^{-} = (R-\mu)I_{(R-\mu)\leq 0}$, R=asset return, $\mu = E(R)$

A portfolio selection problem using semi variance (SV_{μ}) tries to minimize under-performance and does not penalize over-performance with respect to expected return of theportfolio. This risk measure tries to minimize the dispersion of portfolio return from the expected return but only when the former is below the later. To conduct portfolio selection using semi variance, it is not required to compute the covariance matrix; but the joint distribution of securities is needed. If all distribution returns are symmetric or have the same degree of asymmetry, then semi variance and variance produces the same set of efficient portfolios (Markowitz (1959).

When Markowitz (1959) developed his original theory, he did not use the variance as the only measure of risk; he proposed the semi variance as one of the other measures. However, for both theoretical and computational reasons, the use of the variance is the most accepted since it allows, not only a very detailed theoretical analysis of the properties of optimal portfolios (such as the efficient frontier), but also the use of the quadratic optimization methods. Semi variance risk measure is an important

improvement of variance because it only measures the investment return below the expected value. Many models have been built to minimize the semi variance from different angles. Markowitz (1959) recognized the importance of this idea and proposed a downside risk measure known as the semi variance to replace the ordinary variance, since the semi variance is only concerned with the downside, which was the first time that the downside risk had been included in a portfolio selection model. The semi variance measure is more consistent with the perception of the investment risk of a typical investor. However, the attitude towards risks can be vastly different. Since the semi variance is based on the second moment of the downside, it is natural to consider a general *n*th moments of downside to suit different investors. Research on the semi variance did continue in the 1960s and early 1970s. Quirk and Saposnik (1962) demonstrated the theoretical superiority of the semi variance versus the variance. Mao (1970) provided a strong argument that investors will only be interested in downside risk and that the semi variance measure should be used.

Yan and Li (2009) and Yan et al. (2007) substituted variance with semi variance as the risk measure to deal with the multi-period portfolio selection problem. Pınar (2007) also used the downside-risk measure such as semi variance to study the multi-period portfolio selection problem.

2.3.1.2 Absolute Deviation

Konno and Yamazaki (1991) propose a new risk measure called absolute deviation (AD). The purpose of the model is to cope with very large-scale portfolio selection problem because quantifying the deviation from the expected return to make the formula linear instead of a quadratic programming leading to saving in computational time. Konno and Yamazaki (1991) showed that a problem can be solved with more than a thousand securities in a reasonable amount of time. The other advantage is that we do

not have to compute the covariance matrix to do portfolio selection using absolute deviation. In addition, the model generates a portfolio that is quite similar to the mean-variance model if all the returns are normally distributed random variables.

Konno and Yamazaki (1991) showed that the optimal solution using mean-absolute deviation portfolio selection ensure that we do not have to invest in impractically huge number of securities. MAD is easier to compute than Markowitz because it eliminates the need for a covariance matrix. The MV model assumes normality of stock returns, which is not the case; however the MAD model does not make this assumption. The MAD model also minimizes a measure of risk, where the measure in this case is the mean absolute deviation. For a larger mean absolute deviation, the risk is increased. Moreover, MAD is more stable over time than variance as it is less sensitive to outliers and it does not require any assumption on the shape of a distribution. Interestingly, it retains all the positive features of the MV model. MAD is also apply in situations when the number of assets (N) is greater than the number of time periods (T) (Konno & Yamazaki, 1991; Byrne and Lee, 1997, 2004; Brown and Matysiak, 2000; Konno, 2003).

However, the computation time is less significant nowadays due to the advancement of computer. Additionally, the use of MAD is precluded in line with the findings of Simaan (1997) where by the ignorance of the covariance matrix lead to greater estimation risk that outweighs the benefits.

2.3.1.3 Value at risk (VaR) and Conditional Drawdown-at-Risk ($CDaR$)

Value at Risk (VaR) is one of the very popular risk measures widely used in the financial industry. VaR describes the magnitude of likely losses a portfolio can be expected to suffer during normal market movements (Linsmeier and Pearson, 2000). In

plain terms, VaR is a number above which we have only $(1-\alpha)100\%$ of losses and it represents what one can expect to lose with $\alpha\%$ probability, where α is the confidence level.

There are three ways to computeVaR: variance covariance, historical returns and Monte-Carlo simulation. The variance covariance method uses information on the volatility and correlation of stocks to compute the VaR of a portfolio. The Monte-Carlo simulation can be conducted by generating random scenarios for the future returns and computing VaR for these varied scenarios.

To compute VaR using historical returns or any future projected returns of securities, let us assume that we have scenarios of information available to us regarding the future behavior of the returns. Based on this information VaR would be the loss that will be exceeded only by $(1-\alpha)100\%$ of the cases. VaR is derived for losses adjusted for returns using the following approach. Usually losses are in monetary terms, but we list losses in terms of returns (percentage).

Let $V_t = \text{market value at time } t$

$V_{t+h} = \text{market value at time } t+h$

Define Loss L = $\frac{V_t - V_{t+h}}{V_t} = -rx$

The VaR_α satisfies $P(L > \text{Va } R_\alpha) = 1-\alpha$, for a given α (5)

The following non-convex integer program could exactly solve for VaR.

Minimize $VaR = M_{\lfloor\lfloor(1-\alpha)s\rfloor:s\rfloor}(-rx)$

Subject to $x'\mu = E_0$

$$\sum_{i=1}^{n} x_i = 1 \quad , \quad x \geq 0$$

Here the function $M\lfloor k{:}\, N\rfloor$denotes largest k^{th} among the N numbers.

If the portfolio returns are assumed to follow normal distribution, then VaR formulation is a nonlinear programming problem and can be formulated as follows. Suppose there are n securities in which we can invest and their mean return is given by ξ a random variable. Let us suppose that the mean return of the securities ξ has a normal distribution $N(\mu; C)$, where C is positive definite symmetric matrix. Then we can use some of the properties of normal distribution to formulate VaR.

Since $\xi \sim N(\mu, C)$,

$$\text{then } -x'\xi = \sum_{i=1}^{n} -x_i\xi_i \sim N\big(E(X), \sigma(X)\big)$$

Here $E(X) = -x'\mu$ and $\sigma(X) = \sqrt{x'Cx}$. (6)

The following problem can be solved to compute VaR.

Minimize $-(x'\mu) - \phi^{-1}(1-\alpha)\sqrt{x'Cx}$ (7)

Subject to $x'\mu = E_0 \quad , \sum_{i=1}^{n} x_i = 1 \quad , x \geq 0$

VaR is not a coherent measure. As such, risks measured under VaR are not sub-additive or convex. Combining two assets may even increase risks under VaR, which is contrary to the conventional wisdom of diversification. VaR is a point estimate on the tail, which implies it demands a lot more data to get an accurate estimate than variance. Since VaR

is not a convex function of portfolio weights, it is hard to implement its minimization. It can have many local optima that trap the optimization procedure.

Rockafellar and Ursayev (2000) established a new risk measure called Conditional value at risk ($CVaR$). Value at risk measures the minimum loss corresponding to certain worst number of cases but it does not quantify how bad these worst losses are. An investor may need to know the magnitude of these worst losses to discern whether there are possibilities of losing huge sums of money. $CVaR$ quantifies this magnitude and is a measure of the expected loss corresponding to a number of worst cases, depending on the chosen confidence level. Using $CVaR$ makes the portfolio selection problem linear and when we solve it a minimum VaR is found since $CVaR \geq$ VaR (Rockafellar and Ursayev, 2000) but $CVaR$ may have a relatively poor out-of-sample performance compared with VaR if tails are not modeled correctly.

Conditional Drawdown-at-Risk ($CDaR$) is a closely related risk measure to $CVaR$. $CDaR$ was established by Chekhlov et al. (2000) who showed how to implement it for portfolio selection. Portfolio's drawdown on a sample path is the drop of the uncompounded portfolio value as compared to the maximal value attained in the previous moments on the sample path (Krokhmal et al., 2002).

2.3.1.4 Entropy

Entropy is concerned with probabilities as a measure of disorder. It represents the investor's average uncertainty of the returns of a project, and being distribution free, it is not affected by errors due to the fitting of the distribution of returns to a particular distribution. McCauley (2003) argues that entropy has the ability to capture the complexity of the systems without requiring rigid assumptions that can bias the results obtained. Interest in relating entropy to variance dates back to Shannon (1948) who

proposed comparison of continuous random variables according to the entropy power fraction defined as the variance of a Gaussian random variable with given entropy. Shannon (1948) ensures that entropy *H(X),* satisfies some desirable properties of an uncertainty measure (Dionisio et al., 2008).

Let *p(x)* denotes the probability of a random variable *X*. Following Shannon (1948), the entropy of *X* is defined by:

$$H(X) = -\sum_{x\epsilon X} C(x)p(x)\, logp(x)\ ,\ H(X) = E[-C\log p(x)] \qquad (8)$$

Where *C* is some constant. In the above formula, the uncertainty at point *x* is measured as $log\left(\frac{1}{p(x)}\right)$, $p(x) \neq 0$ thus, $H(X)$ is the average uncertainty contained in the variable *X*.

Entropy is a continuous and concave function and is monotonic increasing. For some well-known distribution such as normal, entropy is a function of the variance and so they provide equivalent measure of risk if normality is maintained in the process.

When $C(x)$ is not a constant, but it depends on states/levels of X according to Nawrocki and Harding (1986), the above definition of entropy ignores the structure of the dispersion contain in the frequency classes of a variable. They introduce the state-value weighted entropy especially useful to measure investment risks. The form of weighted entropy is

$$H(X) = -\sum_{x\epsilon X} C(x)p(x)\, logp(x)$$

Two suggested form of *C(x)* are $|s(x) - m|$ and $(s(x) - m)^2$ where $s(x)$ is the state value of frequency classes and m is the mean of the variable.

Entropy is first introduced by Philippatos and Wilson (1972) as a nonparametric alternative measure of portfolio risk to replace variance proposed by Markowitz. So, measuring uncertainty is a way of measuring risk. Their proposed model has two goals that are firstly maximize the expected portfolio return and then to minimize the portfolio entropy.

Philippatos et al. (1972) propose the mean-entropy (ME) model where they use entropies of assets as a measure of risk. They introduce an index based framework where portfolio entropy is computed for a given market index. Suppose, to some extent $R_1, R_2, ..., R_n$ depend on a market index R_I. The mean-entropy (ME) portfolio is then of the form

$$\min_x \sum_{i=1}^{n} x_i^2 H(R_i \mid R_I),$$

$$\text{s.t. } E(w'R) = w'm = \mu_0, \quad w'1_n = 1, \tag{9}$$

where $H(R_i \mid R_I)$ is the conditional entropy of an asset return, R_i, given the market index return, R_I. It should be noted that here conditional entropy, instead of joint entropy, is used to reduce the computational task.

2.4 Multi-objectives portfolio

Single-objective constrained optimization problems are enticing because solution methods are well-known and often only involve concepts from calculus. However, in many real-world scenarios, the single-objective approach proves inadequate. The portfolio optimization problem is one such instance. When creating an investment portfolio, the primary goal for investors is to maximize profit while minimizing risk. Since the return and risk of any investment portfolio are closely interrelated, investors need ways to balance the inherent risk-return trade-off. In recent years portfolio

optimization models consider more criteria than the standard expected return and variance objectives compare widely used Markowitz model. (See Jana.et al., 2009; Arditti, 1967; Konno et al., 1993; Pornchai et al., 1997). However, there is controversy over the issue of whether higher moments should be considered in portfolio selection (see Samuelson, 1970; Arditti and Levy, 1975; Kraus and Litzenberger, 1976; Singleton and Wingender, 1986; Prakash et al., 2003, and Sun and Yan, 2003). Chunhachindaet al., 1997; Arditti, 1967; Arditti and Levy, 1975 assert that higher moments cannot be neglected, unless there is a reason to believe that the asset returns are distributed normally or that higher moments are irrelevant to the investor's decision portfolio.

2.4.1 Mean variance skewness (MVS) portfolio

The mean-variance-skewness (MVS) model

$$Minimize\ x^T V x$$

$$Maximize\ x^T S(x \otimes x)$$

$$\text{Subject to} \qquad x^T M = \mu, \quad \mathrm{x^T}1 = 1 \quad \text{and}\ \mathrm{x_i} \geq 0 \quad \text{for} \quad i = 1,2,\dots,n$$

Prakash et al. (2003), Harvey et al. (2000) and Ibbotson (1975) discuss existence of the higher moments in an asset allocation system if the returns do not follow a symmetrical probability distribution. Moreover, they show that when skewness is included in the decision process, an investor can get a higher return.

The empirical evidence related to the performance of MVS model shows that the incorporation of skewness into MVM can provide significantly better portfolios the non-normal return distributions.

2.4.2 Mean variance entropy portfolio (MVE)

some studies indicate that the portfolio weights obtained from the MV and the MVS can often focus on a few assets or extreme positions (Chunhachinda et al., 1997; Prakash et al., 2003; Bera and Park, 2008), although an important objective of asset allocation is diversification (Bera and Park, 2008; DeMiguel, 2009). In portfolio theory, it is well-known that the diversification reduces unsystematic risk in portfolios. In the other words, the more diversified portfolio weights (probabilities) there are, the more reduced risk there is in the portfolio selection (Dobbins et al., 1994; Gilmore et al., 2005). Diversified portfolios also have lower idiosyncratic volatility than the individual assets (DeMiguel, 2009). Moreover, the portfolio variance decreases as the diversification in portfolio increases. Some authors (Samanta and Roy, 2005; Ke and Zhang, 2008; Jana et. al., 2009; Usta and Kantar, 2011 for instance) utilize entropy of weights together with variance of assets to obtain a diversified portfolio, called as mean-variance-entropy (MVE) portfolio. This model is an extension of MV and is written in the following form:

$$\min_{w} w'V\,w + \xi \sum_{i=1}^{n} w_i \log w_i \,, \tag{10}$$

$$\text{s.t. } E(w'R) = w'm = \mu_0, \;\; w'1_n = 1,$$

where ξ is called momentum factor determining the significance of the term for entropy in the objective function. In MVE, the entropy is not utilized as a measure of risk; rather it is added here to obtain close to uniformly distributed portfolio weights. With proper choice of the momentum factor this model can compromise between the risk and diversification of a portfolio. For $\xi = 0$, MVE equals to MV. Thus, MVE is more general than MV.

2.4.3 Mean variance skewness entropy portfolio (MVSE)

In this approach, an entropy measure is added to the mean-variance-skewness model (MVSM) to generate a well-diversified portfolio that is MVSE. The multi-objective portfolio selection model where investor tries to maximize the skewness of portfolio and entropy of portfolio weights, while simultaneously attempting to minimize the portfolio variance. The multi-objective model based on mean, variance, skewness and entropy can be expressed in the following form:

$$Minimize \quad x^T V x$$

$$Maximize \quad x^T S(x \otimes x)$$

$$Maximize \quad -x^T \ln(x)$$

subject to $x^T M = \mu, \qquad \mathrm{x^T}1 = 1 \quad \text{and} \quad \mathrm{x_i} \geq 0 \quad \text{for} \quad i = 1,2,\dots,n$

They find that the performance of the MVSE portfolio is better than the other models in terms of a variety of portfolio performance measures. Moreover, the MVSE is able to provide smaller portfolio turnover in comparison to the other models; thus, it meaning that the transaction costs associated with the implementation of MVSE are the lowest.

2.5 Portfolio Performance measure

The Sharpe ratio is a commonly used measure of portfolio performance. However, because it is based on the mean-variance theory, it is valid only for either normally distributed returns or quadratic preferences. In other words, the Sharpe ratio is a meaningful measure of portfolio performance when the risk can be adequately measured by standard deviation. When return distributions are non-normal, the Sharpe ratio can lead to misleading conclusions and unsatisfactory paradoxes, see, for example, Hodges

(1998) and Bernardo and Ledoit (2000). For instance, it is well-known that the distribution of hedge fund returns deviates significantly from normality (see, for example, Brooks and Kat, 2002; Agarwal and Naik, 2004; Malkiel and Saha, 2005); therefore, performance evaluation of hedge funds using the Sharpe ratio seems to be dubious. Moreover, recently a number of papers have shown that the Sharpe ratio is prone to manipulation (see, for example, Leland, 1999; Spurgin, 2001; Goetzmann et al., 2002; Ingersoll et al., 2007). Manipulation of the Sharpe ratio consists largely in selling the upside return potential, thus creating a distribution with high left-tail risk.

There are vast literatures on performance evaluation that takes into account higher moments of distribution is. Motivated by a common interpretation of the Sharpe ratio as a reward-to-risk ratio, many researchers replace the standard deviation in the Sharpe ratio by an alternative risk measure. For example, Sortino and Price (1994) replace standard deviation by downside deviation.

In order to evaluate the performance of portfolio models, a number of alternative performance measures have been proposed in the literature. As a traditional performance measure, the Sharpe ratio (SR) has been used extensively and its formula is given as the following general form

$$SR = \frac{E[R_p]}{\sqrt{\sigma^2[R_p]}}$$

where R_p is the return of portfolio.

However, since the SR is based on the mean-variance theory, it is only valid for normally distributed returns. Particularly, the SR can lead to misleading conclusions when the return distributions are skewed or display heavy tails. Several alternatives to the SR for optimal portfolio selection have been proposed in the literature. Some of

these alternatives are presented in the following: The adjusted for skewness Sharpe ratio (ASR) (Zakamouline, 2009), which takes into accounts the skewness of portfolio, is defined as follows:

$$ASR = SR\sqrt{1 + \frac{Sk[R_p]}{3}SR}$$

(i) The mean absolute deviation ratio, (MADR) (Konno, 1990), which considers the risk as mean absolute deviation, is given as follows:

$$ADR = \frac{E[R_p]}{E[|R_p - ER_p|]}$$

(ii) The Sortino-Satchell ratio (SSR) and Farinelli and Tibiletti ratio (FTR) (Farinelli et al., 2009), are performance measures based on the partial moments and their formulas are given as follows, respectively:

$$SSR = \frac{E[R_p]}{\sqrt{E[\max(-R_p,0)^2]}},$$

where $E[\max(-R_p, 0)^2]$ is the lower partial moment of order 2.

(iii) $$FTR = \frac{\sqrt[u]{E[\max(R_p,0)^u]}}{\sqrt[v]{E[\max(-R_p,0)^v]}} \quad u, v > 0,$$

where $E[\max(-R_p, 0)^v]$ and $E[\max(R_p, 0)^u]$ are the lower partial moment of order *v* and the upper partial moment of order *u*, respectively. The selection of $u$and $v$are associated to investors' styles or preferences. In the empirical part in chapter 3, we will consider the following cases for *u* and *v* according to Farinelli et al. (2008) and Keating and Shadwick (2002) with$u = 0.5$, $v = 2$ for a defensive investor; $u = 1.5$, $v = 2$ for

a conservative investor; $u = 1$, $v = 1$ for a moderate investor. Additionally, it is known that if $u = 1$, $v = 1$, the FTR reduces to the Omega ratio of Keating (2002).

CHAPTER 3: COMPARISON BETWEEN VARIANCE AND ENTROPY

3.1 Introduction

Variance and other indices continue to be popular because of simplicity. The historical development has resulted in variance playing the central role in measuring dispersion, uncertainty, evaluating fit, and many more. While variance measures compactness of data around the mean, entropy, on the other hand measures diffuseness of the density irrespective of the location of compactness. Entropy like variance is a collective measure of uncertainty, but unlike variance, it can be either a cardinal or ordinal variable.

3.2 Entropy at a glance

Entropy measures the uncertainty inherent in the distribution of a random variable. Suppose $p(x)$ be the probability of a random variable X. Following Shannon (1948), the entropy of X is defined by:

$$H(X) = -\sum_{x\epsilon X} p(x)\, log\, p(x) \qquad (3.1)$$

In the above formula, the uncertainty at point *x* is measure as $log\frac{1}{p(x)}$ so that $H(X)$ is the average uncertainty contained in the variable X. Entropy is non-negative. If the outcome is certain, the entropy is zero and it is positive when the outcome is not certain. Entropy is concave and continuous function. When the values of some probabilities are changed by small amount, the entropy should also change by only a small amount. In finance, entropy is used as a synonym for risk in the sense that uncertainty causes loss, and so, measuring uncertainty is an alternative way of measuring risk. Philippatos and Wilson (1972) use Shannon entropy as a measure of risk of securities. The logic of their

approach was that risk inherent in an investment whose returns are uncertain is adequately captured by the dispersion in the probabilities of the returns.

The joint entropy and the conditional entropy are simple extensions that measure the uncertainty respectively, in the joint distribution and the uncertainty in the conditional distribution of a pair of random variables. The joint entropy $H(x,y)$ of a pair of discrete random variables with a joint distribution $p(x,y)$ is defined as:

$$H(X,Y) = -\sum_{x\epsilon X}\sum_{y\epsilon Y} p(x,y)\log\ p(x,y) \tag{3.2}$$

Similarly, the conditional entropy $H(Y/X)$ is defined as:

$$H(Y|X) = -\sum_{x\epsilon X}\sum_{y\epsilon Y} p(x,y)\log\ p(y|x) \tag{3.3}$$

As noted by Nawrocki and Harding (1986), the above definition of entropy ignores the structure of the dispersion contain in the frequency classes of a variable. They introduce the state-value weighted entropy that is especially useful to measure investment risks. The form of weighted entropy is

$$H(X) = -\sum_{x\epsilon X} C(x)p(x)\ log\ p(x) \tag{3.4}$$

Here *C(x)* is not a constant, but it depends on states levels of X. One suggested form of *C(x) is*$|s(x)-m|$, where $s(x)$ is the state value of frequency classes and m is the mean of the variable.

Philippatos and Gressis (1975) conclude that if the asset distribution is either normal or uniform, mean-entropy and mean-variance portfolios are equivalent. Ebrahimi et al. (1999) examined the position of variance and entropy in ordering distributions and random prospects and argue that in terms of ordering distributions these measures do

not show any universal relationship. Using a Legendre series expansion, they show that entropy is a function of not only variance but also higher-order moments of a random variable. However, when continuous variables are transformed, under certain conditions, the order of the variance and entropy is similar. These authors conclude that, entropy uses more information than variance and thus, it offers better characterization of $p_x(X)$. Ebrahimi et al. (1999) provides significant insights about entropy and its relation to variance and higher-order moments by approximating the density function through a Legendre series expansion function. A smooth and continuous density can be well approximated as

$$p(x) \approx a_0 G_0(x) + a_1 G_1(x) + \cdots + a_N G_N(x)\,, \tag{3.5}$$

where $G_i(x), \quad i = 1, \ldots, N$ are Legendre polynomials:

$$G_0(x) = 1, \quad G_1(x) = x, \qquad G_2(x) = 0.5(3x^2 - 1), \ldots \quad .$$

Note that

$$\int_{-1}^{1} G_i(x) G_j(x) dx = \frac{2\delta_{ij}}{2i+1},$$

where δ_{ij} is the Kronecker's delta, and $x\epsilon[-1, +1]$. One might obtain a_0 and a_1 to satisfy the normalization restriction and mean zero restriction.

Since

$$x^2 = \frac{1}{3}[2G_2(x) + G_0(x)],$$

variance is approximated by

$$\mathrm{V(x)} = \int x^2\, p(x) dx \approx \tfrac{1}{3}\left[\tfrac{4}{5} a_2 + 2a_0\right]$$

This approximation reveals that the variance increases if and only if a_2 increases. Other $a_i,\ i \geq 3$, do not influence the variance.

Now, employing Eq. (3.1), it can be verified that the derivative of *H* with respect to a_2 is

$$\frac{\partial H}{\partial a_2} \approx -\int G_2(x) \log[a_0 G_0(x) + a_1 G_1(x) + \cdots + a_N G_N(x)] dx \,.$$

Entropy increases with variance if this expression is positive, and also the variation of entropy depends on many more parameters than just a_2. It is revealed fromthe Legendre series expansion that entropy is connected to higher order moments of a distribution, which unlike the variance, could provide a much improved characterization of $p(x)$. Maasoumi and Racine (2002) argue that in the case of unknown probability distribution, the entropy formulate an alternative statistical measure for the uncertainty**,** predictability and goodness-of-fit. The entropy can be replaced by the variance only in the case of Gaussian distributions. The 'fat' tailed distributions are not fully described by a variance; in such a case, we need more parameters. When the distribution is known, entropy can be calculated from variance in most of the cases. To make this clear we listed entropies and variances of some well-known distributions in Table 3.2. It is obvious that entropy is a function of variance (if it exists) and so if the form of the distributions is known, use of either entropy or variance is equivalent.

The standard-deviation and the entropy usually decrease when we include one more asset in the portfolio (Dionosio, 2005). This fact allows us to figure out that entropy is responsive to the effect of diversification. These results can be explained by the fact that when the number of assets in the portfolio increases, the number of possible states of the system (portfolio) declines progressively and the uncertainty about that portfolio tends to fall. Since maximizing Shannon’s entropy subject to some moment constraints

implies estimating weight that is the closest to the uniform distribution (i.e., equally weighted portfolio), well-diversified optimal portfolio can be achieved.

Based on the above discussion, we can conclude that

1. variance is easy to calculate and more familiar than entropy;
2. entropy and variance are equivalent for normal or uniform distribution;
3. entropy is more informative than variance;
4. entropy can be estimated nonparametrically; thus, unlike variance, entropy is not restricted to symmetric and normal distribution;
5. like variance, portfolio entropy is sensitive to diversity

3.3 Estimation

Beirlant (1997) categorize entropy estimation from the real data into three basic methods: plug-in estimate, sample spacing estimate and nearest neighbor distance estimate.

(1) Plug-in estimates: There are four approaches for plug in estimates of entropy

(a) Integral estimate: An integral estimate of entropy is the sample version of equation (3.2) and has the form

$$H_n = -\sum_{A_n} f_n(x) log\, f_n(x) dx \qquad (3.6)$$

where $f_n(x)$ is a consistent density estimate evaluated at a bounded set A_n that typically exclude that small or tail values of f_n to make sense of $-lnf_n$. Dmitriev and Tarasenko (1973) propose to estimate entropy with this formula by plug-in kernel density estimator and show that is strongly consistent. Joe (1989) considers entropy in multivariate case where he points out that the calculation of the density by kernel estimator is difficult for more than two variables; however, it provide good estimate for

low dimensional data. Györfi and van der Meulen (1987) use histogram based density estimator to compute entropy and show that it is strongly consistent for finite entropy.

(b) Resubstitution estimate: A resubstitution estimate of entropy has the form

$$H_n = -\frac{1}{n}\sum_{i=1}^{n} ln f_n\,(X_i) \tag{3.7}$$

Ahmad and Lin (1976) propose using kernel density estimate in this formula of entropy and show its mean square consistency. Joe (1989) finds the asymptotic bias and variance of this estimator and noted that as the dimension of the data increases, sample size should be large enough to obtain reasonable estimate. Hall and Morton (1996) show that histogram-based resubstitution estimator is root-n consistent for one dimensional data but for two-dimensional data it has significant bias. They also show that under certain condition this estimator with kernel density is *root-n* consistent.

(c) Splitting data estimate: Suppose $X_1, \dots, X_l$ and $X_1^*, \dots, X_m^*$ are two subsamples of $X_1, \dots, X_n$ with $l + m = n$ and f_l be a density estimate based on $X_1, \dots, X_l$ then a splitting data estimate of entropy has the form

$$H_n = -\frac{1}{m}\sum_{i=1}^{m} I_{[X_i^* \in A_l]}\, ln f_l(X_i^*) \tag{3.8}$$

Györfi and van der Meulen (1987, 1989) use histogram and kernel density estimates for f_l under some mild tail and smoothness conditions on f_l this estimator is strongly consistent.

(d) Cross validation estimate: Ivanov and Rozhkova (1981) propose using the resubstitution formula with a kernel density estimate base on cross validation or leave-one-out:

$$H_n = -\frac{1}{n}\sum_{i=1}^{n} I_{[x_i \in A_n]}\, ln f_{n,i}(X_i) \tag{3.9}$$

They show that this estimator is strongly consistent. Hall and Morton (1993) show that under certain conditions it provides *root-n* consistent estimate for one to three-dimensional data.

(2) Sample spacing estimate: The sample spacing method is to estimate the density using sample spacing method and then use either integral estimate or resubstitution estimate to compute entropy. Let the ordered observations be $X_{n,1} \leq X_{n,2} \leq \cdots \leq X_{n,n}$. The sample spacing method is to estimate the density using spaces between ordered observations $X_{n,i+m} - X_{n,i} (1 \leq i < i + m \leq n)$:

$$f_n(x) = \frac{m}{n} \frac{1}{X_{n,im} - X_{n,(i-1)m}} \tag{3.10}$$

The entropy can then be computed using the formula in equation (3.6) or (3.7). Tarasenko (1968), Beirlant and Zuijlen (1985)and Hall (1984) find that this estimator is weakly consistent and asymptotically normal.

(3) Nearest neighbor distance estimate: Kozachenko and Leonenko (1987) propose a formula for estimating entropy using the nearest neighbor distance of observations $\rho_{n,i}$ defined as $\rho_{n,i} = min_{j \neq i, j \leq n} \|X_i - X_j\|$. Then the nearest neighbor estimate is

$$H_n = \frac{1}{n} \sum_{i=1}^{n} \ln(n \rho_{n,i}) + \ln 2 + C_E \tag{3.11}$$

where C_E is the Euler constant: $C_E = -\int_0^\infty e^{-t} \ln t \, dt$

Tsybakov and van der Meulen (1994) show that this estimator is root-n consistent.

3.3.1 Entropy estimation from sample data

For a known density function, entropy can be calculated in terms of the estimated parameters (Cover and Thomas, 1991).Arellano-valle and Richter (2012) provides a

general expression for the entropy of multivariate skew elliptical distributions. A Bayesians parametric estimation of entropy is proposed by Gupta and Srivastava (2010). R codes for calculating entropy for a given density with different methods are given in Appendix 1. However, we seldom know the true density of the data in hand. Vasicek (1976) estimates entropy directly from a given set of data based on sample spacing. Correa (1995) modified Vesicek's estimator which offers smaller mean square error. The common practice of computing entropy is to first estimate the density using histogram or kernel density methods (Hall and Morton, 1996; Moddemeijer, 1999; Darbellay and I. Vajda, 1999) and subsequently plug-in the raw estimate of the probability, $p(x)$in equation (1) or (2). Plug-in estimators using histogram and kernel density provide consistent entropy estimates for low dimensional data (Györfi and van der Meulen, 1987).

3.3.1.1 Histogram

Widely employed in exploratory data analysis, a histogram is usually a graphical representation of the frequency distribution of a dataset. Because of the ease and simplicity of structure and interpretation, histograms are still popular compare to more sophisticated kernel–based density estimators (Wand,1994; Simonoff and Udina,1997).

Summary quantities such as entropy using histograms, however, the values of such quantities depend upon the number of bins used (or the bin width used) and the location of the bins (Knuth, 2006). Let $X = \{X_1, X_2, \dots, X_n\}$ be a univariate dataset with probability density function $f(x)$. Martinez andMartinez (2007) describe the construction of a histogram at first it needs an origin for the bins t_0 (also referred to as the anchor) and a bin width h. Selection of these two parameters defines a mesh (position of all the bins) over which the histogram will be constructed. Each bin is

represented by a pair of bin edges as $B_k = [t_k, t_{k+1}]$, where $t_{k+1} - t_k = h$ for all k. Let c_k(bin count for B_k) is the number of observations in B_k:

$$c_k = \sum_{i=1}^{n} I_{B_k}(x_i)$$

where I_{B_k} is defined as

$$I_{B_k}(x_i) = \begin{Bmatrix} 1 & x_i in\ B_k \\ 0 & x_i not\ in\ B_k \end{Bmatrix}$$

while the density estimate for the underlying population (c_k for all k) satisfies the non–negativity condition necessary for it to be a bona fide probability density function, the summation of all the probabilities do not necessarily add to unity. To satisfy that condition, the probability density function estimate, $\hat{f}(x)$, as obtained from a histogram, is defined as:

$$\hat{f}(x) = \frac{c_k}{nh} \text{ for } x \text{ in } B_k$$

This assures that $\int \hat{f}(x)\, dx = 1$ is satisfied, and $\hat{f}(x)$, represents a valid estimate for the probability density function of the population underlying the dataset.

The usual practice of histogram construction is using $t_0 = \min(X)$. Wand and Jones (1994) noted that the value of c_k heavily depends upon the parameters t_0 and h. Simonoff and Udina (1997) provide a method to quantify the effects of changing the parameter $t_0$during the construction of a histogram. A common method to determine bin width h is:

$$h = \frac{\max(X) - min(X)}{m}$$

Using a small value for m in a large bin width causes a histogram that offers the shape of the underlying distribution impractical; and using large value for bins in a small bin width produces a histogram that capture the shape of the underlying distribution extremely noisy. Unless the underlying population distribution is Uniform by considering single number of bin $(m = 1)$, and information relating to shape, modality, and symmetry will be lost. Even if we consider the number of bins more than one, there will still be a loss of information relating to shape, modality, and symmetry. Since, within each bin it is considered that observations are uniformly distributed. Examples of these two extreme cases suggests that "optimal" number of bins should be used to construct a histogram that can effectively capture information relating to shape, modality, and symmetry and imply sufficient values for summary quantities. Knuth (2006) suggests that the number of bins should be sufficient enough to capture all major structures of the underlying distribution, but small enough to avoid finer details and random sample noise. Thus, to achieve a proper balance between "degree of detail" and "noisy-ness" for a given dataset in selecting an "optimal" number of bins for constructing a histogram, is a sophisticated task.

Methods for the number of bins selection

Robust estimation of entropy and mutual information from histograms is a challenging task. Perhaps the earliest reported method for constructing histograms is provided by Sturges, 1926. It is based on the assumption that a good distribution should have binomial coefficients $\binom{m-1}{i}$, $i = 0, 1, 2, \ldots, m-1$ as its bin counts. With suggested bin width $= 1 + \log_2 n$, the bin number can then be determined by $m = \frac{R}{w}$, where R is the range of the dataset. The Sturges' rule assumed that the data are normally distributed, thus, is not suitable for non-normal data.

Scott (1979) gave a formula for the optimal histogram bin width which asymptotically minimizes the integrated mean squared error (IMSE). The IMSE is defined as:

$$IMSE = \int MSE(x)dx$$

$$= \int E(\hat{f}(x) - f(x))^2 dx$$

where $\hat{f}(x)$estimated density using histogram, and $f(x)$ is the actual probability density of the underlying population. By considering Gaussian as the actual probability density, using this error metric, Scott (1979) suggests the bin width to be used as:

$$h = \frac{3.49s}{n^{\frac{1}{3}}}$$

where s is the sample standard deviation.

Freedman and Diaconis (1981) make a slight modification with this formula and suggest:

$$h = \frac{2(IQR(X))}{n^{\frac{1}{3}}}$$

where IQR(X) is the Inter–Quartile Range for the dataset X.

Knuth (2006) criticize these popular methods since the certain assumptions about the underlying population for estimating the value of MISE, which may not be satisfied by all datasets. The author proposes for using of Bayesian approach to select the number of bins.

There is substantial literature on how to select the number of bins; see, for example, He and Meeden (1997), who provide a decision theoretic approach to the selection of

the number of bins. Several authors have addressed histogram as a density estimator; see, as examples, Sturges (1926), Doane (1976), Scott (1979), Freedman and Diaconis (1981), Rudemo (1982),Stone (1985), Kanazawa (1992), Wand (1996; 1997), Shimazaki and Shinomoto (2007), Scott and Scott (2008), Wang and Zhang (2012), and Lu et al. (2013). Stone (1985) offers a procedure based on minimization of a loss function defined on the basis of bin probabilities and number of bins. Rudemo (1982) proposes a method based on Kullback-Leibler risk function and cross–validation techniques. Wand (1996) extends Scott's method to have good large sample consistency properties. The use of Akaike's Information Criterion (AIC) and Kullback-Liebler Cross Validation techniques for preparing histogram investigated by Hall (1990). To construct histogram, Birge et al. (2006) use risk function based on a penalized maximum likelihood. Assuming that the data are sampled independently Shimazaki et al. (2007) propose minimization of the estimated cost function based on a modified MISE.

Recently Hacine-Gharbi et al. (2013) derive a new approach for estimating the optimal number of histogram bins by minimizing the MSE for estimation of entropy and empirically shows its better performance over Sturges, Scott and Freedman-Diaconis rules. They proposed approximating pdfs with histogram by reducing the bias and MSE for estimating mutual information. When using the histogram approach with discrete finite-sample data, two biases appear: the R_{bias} caused by insufficient representation of the pdfs using the histogram, and the N_{bias} due to the finite sample size (Pearce and Hirsch, 2000). This Low MSE (LMSE) MI estimation approach is explained below in detail.

For histogram-based mutual information (MI) estimation, Pearce and Hirsch divide the XY-plane into $k_x \times k_y$ equally sized $\Delta_x \times \Delta_y$ cells. In approximation of MI

$$I_2(X;Y) \approx \sum_{j=1}^{k_y}\sum_{i=1}^{k_x} p_{ij} \log \frac{p_{ij}}{p_i p_j}$$

for two Gaussian variables X and Y, with standard deviation ∂_x and ∂_y respectively, and a correlation coefficient ρ, the first order R_{bias} with Taylor series expansion can be expressed as:

$$R_{bias} = \frac{\rho^2}{24(1-\rho^2)}\left(\left(\frac{\Delta_X}{\sigma_x}\right)^2 + \left(\frac{\Delta_Y}{\sigma_y}\right)^2\right)$$

By Counting the number k_{ij} of samples in cell (i,j), a classical estimator of p_{ij}, for a total number of samples N, can be obtained as $\hat{p}_{ij} = \frac{k_{ij}}{N}$ the rows and columns are then summed to obtain $\hat{p}_i = \sum_{j-1}^{k_y} \hat{p}_{ij}$ and $\hat{p}_j = \sum_{i-1}^{k_x} \hat{p}_{ij}$.

The expression of the MI becomes:

$$\hat{I}_2(X;Y) = \sum_{j=1}^{k_y}\sum_{i=1}^{k_x} (\frac{k_{ij}}{N}) \log(\frac{k_{ij}N}{k_i k_j})$$

A Taylor expansion around k_{ij} leads to the first order N_{bias}

$$N_{bias} = \frac{(k_X - 1)(k_Y - 1)}{2N}$$

The total first order bias is the sum of the R_{bias} with the N_{bias} leads to:

$$B_{I_{XY}} = E\left(\hat{I}_{XY}\right) - I_{XY} \approx \frac{(k_X - 1)(k_Y - 1)}{2N} - \frac{\rho^2}{24(1-\rho^2)}\left(\left(\frac{\Delta_X}{\sigma_x}\right)^2 + \left(\frac{\Delta_Y}{\sigma_y}\right)^2\right)$$

Let A_x and A_y be the extents of X and Y respectively. Then

$k_x\Delta_X = A_x$ and $k_y\Delta_Y = A_y$

$\alpha_x.\sigma_x = A_x$ and $\alpha_y.\sigma_y = A_y$

where α_x and α_y are constants, and if we assume $k = k_x = k_y$ for simplicity, then we can write:

$\frac{\Delta_X}{\sigma_x} = \frac{\alpha_x}{k}$ and $\frac{\Delta_Y}{\sigma_y} = \frac{\alpha_y}{k}$

Finally,

$$B_{I_{xy}} \approx \frac{(k-1)^2}{2N} - \frac{\rho^2}{24(1-\rho^2)}\left(\left(\frac{\alpha_x}{k}\right)^2 + \left(\frac{\alpha_y}{k}\right)^2\right)$$

Furthermore, it has been shown previously (Scott, 1992) that the variance of $\hat{I}_2(X;Y)$ for two Gaussian variables X, Y can be approximated by:

$$var_{I_{xy}} \approx \frac{\rho^2}{N}$$

Thus, MSE of $\hat{I}_2(X;Y)$ is defined as:

$$MSE_{I_{xy}} = var_{I_{xy}} + (B_{I_{xy}})^2$$

$$\approx \frac{\rho^2}{N} + \left(\frac{(k-1)^2}{2.N} - \frac{\rho^2}{24(1-\rho^2)}\left(\left(\frac{\alpha_x}{k}\right)^2 + \left(\frac{\alpha_y}{k}\right)^2\right)\right)^2$$

The optimal number of bins for histogram, k_{opt}, can then be obtained by minimizing the MSE:

$$k_{opt} = \arg\min_k MSE_{I_{xy}}$$

$$= \arg\min_k \left[\frac{\rho^2}{N} + \left(\frac{(k-1)^2}{2.N} - \frac{\rho^2}{24(1-\rho^2)} \left(\left(\frac{\alpha_x}{k}\right)^2 + \left(\frac{\alpha_y}{k}\right)^2 \right) \right)^2 \right]$$

$$= \arg\min_k \left[\left(\frac{(k-1)^2}{2.N} - \frac{\rho^2}{24(1-\rho^2)} \left(\left(\frac{\alpha_x}{k}\right)^2 + \left(\frac{\alpha_y}{k}\right)^2 \right) \right)^2 \right]$$

$$= \arg\min_k \left[(B_{I_{xy}})^2 \right]$$

Therefore, the bin number which minimizes $(B_{I_{xy}})^2$ is the same as the one which minimizes $MSE_{I_{xy}}$. Then, k_{opt} is determined by the solution of the following equation:

$$\frac{(k-1)^2}{2.N} - \frac{\rho^2}{24(1-\rho^2)} \left(\left(\frac{\alpha_x}{k}\right)^2 + \left(\frac{\alpha_y}{k}\right)^2 \right) = 0$$

Using simple algebraic manipulations, we rewrite the above expression as:

$$k^4 - 2k^3 + k^2 - L = 0$$

with constant

$$L = \frac{N\rho^2}{12(1-\rho^2)} (\alpha_x{}^2 + \alpha_y{}^2)$$

$$= \frac{N\rho^2}{12(1-\rho^2)} \left((\frac{A_x}{\sigma_x})^2 + (\frac{A_x}{\sigma_y})^2 \right)$$

The number of bins as the nearest integer value of the positive real solution is:

$$k = round \left\{ \frac{1}{2} + \frac{1}{2} \sqrt{1 + 4\sqrt{L}} \right\}$$

Traditionally, the histogram for a Gaussian distribution can be defined on a 6σ range which leads to $A_x = 6\sigma_x$ and $A_y = 6\sigma_y$. Hence, the number of bins for the proposed low MSE histogram-based MI estimator becomes:

$$k = round\left\{\frac{1}{2} + \frac{1}{2}\sqrt{1 + 4\sqrt{\frac{6N\hat{\rho}^2}{1-\hat{\rho}^2}}}\right\}$$

where the unknown correlation coefficient ρ has been replaced by its classical estimator $\hat{\rho}$, and round stands for the closest integer of a real variable.

In a case of estimating the mutual information between continuous random variables and a discrete class, the MI is expressed as a function of the entropy and the conditional entropy:

$$I_2(X;C) = H(x) - \sum_{c\epsilon C} p_c\,(c) H(X|C=c)$$

where p_c is just the probability that variable *C* takes the value *c*.

A low MSE estimation of $I_2(X;C)$ can be achieved by a low MSE estimation of the entropy $H(x)$, and that of each of the conditional entropies $H(X|C=c)$ for the individual classes c's.

The discrete approximation of $H(x)$ can be written:

$$\widehat{H}_x = -\sum_{i=1}^{I}(\frac{k_i}{N})\log(\frac{k_i}{N}) + \log(\Delta x)$$

k_i = number of observations in bin *i*

The total first order bias can be derived for $\widehat{H}_x$:

$$B_{\hat{H}_x} = E(\hat{H}_x) - H_x \approx \frac{1}{24}\left(\frac{\Delta x}{\sigma_x}\right)^2 - \frac{k-1}{2N}$$

The variances of the estimator, like for the MI estimator, is independent of the number of bins and is expressed as

$$var_{\hat{H}_x} \approx \frac{1}{2N}$$

The MSE of $\hat{H}(X)$ is defined as:

$$MSE_{\hat{H}_x} = var_{\hat{H}_x} + (B_{\hat{H}_x})^2$$

$$\approx \frac{1}{2N} + (\frac{1}{24}\left(\frac{\alpha_x}{k}\right)^2 - \frac{k-1}{2N})^2$$

The optimal number of bins in the sense of a low MSE estimator of entropy H_x is defined as:

$$k_{xopt} = \arg\min_k MSE_{\hat{H}_x}$$

$$= \arg\min_k \left[\frac{1}{2N} + \left(\frac{1}{24}\left(\frac{\alpha_x}{k}\right)^2 - \frac{k-1}{2N}\right)^2\right]$$

$$= \arg\min_k \left[\left(\frac{1}{24}\left(\frac{\alpha_x}{k}\right)^2 - \frac{k-1}{2N}\right)^2\right]$$

$$= \arg\min_k \left[(B_{\hat{H}_x})^2\right]$$

Therefore, the bin number which minimize the bias $(B_{\hat{H}_x})^2$ is the same as the one which minimizes the $MSE_{\hat{H}_x}$. The optimal value for k is obtained by solving:

$$k^3 - k^2 - G = 0$$

where constant G is given as:

$$G = \frac{N\alpha_x^2}{12} = \frac{NA_x^2}{12.\sigma_x^2}$$

The solution is given by

$$k = round\left\{\frac{\xi}{6} + \frac{2}{3\xi} + \frac{1}{3}\right\}$$

With

$$\xi = \sqrt[3]{(8 + 108G + 12\sqrt[2]{12G + 81G^2})}$$

Traditionally, the histogram for a Gaussian distribution is defined on a 6σ range leading to $A_x = 6\sigma_x$.

In that case $\xi = \sqrt[3]{(8 + 324N + 12\sqrt[2]{36N + 729N^2})}$.

3.3.1.2 Kernel density estimation

Kernel density estimation is a generalization of histogram based method and is nonparametric. The most important part of kernel density estimation is the bandwidth selection. Joe (1989) finds that optimum bandwidth selection for entropy is quite difficult as the dimension increases. Here, we estimate the entropy in two steps. We first estimate the kernel density with an optimal bandwidth and then calculate entropy from the estimated density. If we have a good density estimate, our entropy computation will have less error. In the next subsection, we discuss kernel density estimator and its implementation in real data using R package *ks*.

For a d-variate random sample $x_1, x_2, \ldots., x_n$ drawn from a density f, the kernel density estimate is defined by

$$\hat{f}(x; H) = n^{-1} \sum_{i=1}^{n} K_H (x - X_i) \tag{3.12}$$

where $x = (x_1, x_2, \ldots, x_d)^T$ and $X_i = (X_{i1}, X_{i2}, \ldots, X_{id})^T$, $i = 1,2, \ldots, n$. Here $K(x)$ is the kernel which is usually a symmetric probability density function, $K(x) = (2\pi)^{-\frac{d}{2}} \exp\left(-\frac{1}{2} x^T x\right)$, the standard normal for example. H is known as the bandwidth which should be symmetric and positive-definite. Earlier studies (Wand and Jones, 1993 and Simonoff, 1996 for instance) show that the choice of *K* is not crucial; however, the performance of $\hat{f}$ strongly depends on the choice of H.

Wand and Jones (1994) extend the idea of univariate plug-in method for bandwidth selection to multivariate kernel density estimation and show that it has a good rate of convergence compared to other methods of bandwidth selection. The choice of *H* is usually based on minimization of some global error criterion. The simplest criterion to work with is mean integrated squared error $(MISE)$ given by:

$$MISE\{\hat{f}(.; H)\} = E \int \{\hat{f}(x; H) - f(x)\}^2 \, dx \tag{3.13}$$

It is obvious that the optimal bandwidth H_{MISE} does not have a closed form. The useful approximation to $MISE\{\hat{f}(.; H)\}$ is the asymptotic $(MISE)(AMISE)$ of $\hat{f}(.; H)$ given by

$$AMISE\{\hat{f}(.; H)\} = n^{-1}|H|^{-\frac{1}{2}} R(K) + \frac{1}{4}\mu_2(K)^2 (vechH)^T \Psi_F (\text{vechH}) \tag{3.14}$$

Here $R(K) = \int K(x)^2 \, dx$ and $\mu_2(K) = \int x_i^2 \, K(x) dx < \infty$.

Also $\Psi_F = \int vech\{2H_f(x) - dgH_f(x)\} \left[vech\{2H_f(x) - dgH_f(x)\}^T dx\right]$

where $vech$ is the vector half operator. $H_f(x)$ is the hessian matrix of f. The notation *dg* denotes the diagonal matrix form by replacing all off-diagonal entries by zeroes.

Wand (1992) shows that if the entries of $H_f(x)$ are continuous and square integrable and all entries of H as well as $n^{-1}|H|^{-\frac{1}{2}}$ tend to zero as $n \to \infty$ then

$$MISE\{\hat{f}(.;H)\} = AMISE\{\hat{f}(.;H)\} + o\left\{n^{-1}|H|^{-\frac{1}{2}} + tr^2(H)\right\} \quad (3.15)$$

Wand and Jones (1994) demonstrate the way of getting optimal diagonal plug-in estimate of H by minimizing $AMISE\{\hat{f}(.;H)\}$. They also show that the rate of convergence of this estimator $n^{\frac{-5}{4}}$ when *d=1*, and it is $n^{\frac{-2}{d+4}}$ for $d \geq 2$.

Duong and Hazelton (2003) argue that estimation of full bandwidth matrix by plug-in method instead of diagonal has advantages compared to existing method. It produces a finite bandwidth matrix and requires significantly fewer pilot bandwidths. They also provide the algorithm for doing this and implemented it in *ks* package in R. An example is given below that shows how we can use *ks* package to select optimal plug-in bandwidth selector.

```
# simulate from bivariate skew normal distribution

library(sn)

n=1000          # set size of random sample is 1000

m=c(0.0, 0.0)       # set mean vector

Omega = diag(2)   # set variance-covariance matrix

alpha = c(0.5,0.4) # set shape parameter
```

```
#generate random sample of size n with parameter m, Omega and alpha

X =rmsn(n, m, Omega, alpha)

# plug-in bandwidth selections

Library(ks)

# full bandwidth matrix

bw1=Hpi(X, pilot="amse") # Wand & Jones (1994)

bw2=Hpi(X, pilot="samse") # Duong & Hazelton (2003)
```

The function Hpi will give unconstrained plug-in selectors (full bandwidth matrix) and Hpi.diag will give diagonal plug-in selectors.

```
# diagonal bandwidth matrix

bw3=Hpi.diag(X, pilot="amse")

bw4=Hpi.diag(X, pilot="samse")
```

"amse" or "samse" represent the type of pilot estimation. There are three other arguments which further specify the plug-in selector: *nstage* is the number of pilot estimation stages (1 or 2). Wand and Jones (1994) recommend for using two-stage pilot estimation. The argument *pre* involves the pre-transformations ("scale" or "sphere"). If pre sphere is set, observations are transformed before performing bandwidth selector algorithm as

$$X^* = S^{-\frac{1}{2}}X,$$

where S = sample covariance matrix of the untransformed data.

If “scale” is selected, data is transformed as

$$X^* = S_D^{-\frac{1}{2}} X,$$

where $S_D = diag(s_1^2, s_2^2, \dots)$ and $s_1^2, s_2^2, \dots$ are the diagonal elements of S.

The options *pre-sphering* or *pre-scaling* is only for the unconstrained bandwidths. For the diagonal bandwidths, only the *pre-scaling* can be used to avoid the back-transformation of *pre-sphering* results in a non-diagonal matrix.

Rudemo (1982) and Bowman (1984) implement cross validation method for selecting the smoothing parameter in the kernel density estimation while a biased cross-validation method is proposed by Scott and Terrell (1987) for kernel density Taylor (1989) proposes the use of bootstrap method Sain et.al. (1994) and compare the earlier literature and derive multivariate kernel density estimation using the product kernel estimate. Their simulation studies suggest that the biased cross-validation method of Scott and Terrell (1987) performs well with a little variation as compared to the other two methods. Duong and Hazelton (2005) developed a smooth cross validation (SCV) method for multivariate data and show that it has a better convergence rate compare to other cross validation techniques like unbiased cross validation (UCV) and biased cross validation (BCV) and comparable with plug in bandwidth selector. The ks package of R (Doung, 2007) is able to compute multivariate optimal bandwidth using different methods of bandwidth selections:

bw5 = Hlscv(x = data) # Bowman (1984) and Rudemo (1982)

bw6 =Hlscv.diag(x = data)

bw7 = Hbcv(x = data) # Sain, Baggerly& Scott (1994)

bw8 = Hbcv.diag(x = data)

bw9 = Hscv(x = data) # Duong & Hazelton (2005)

bw10 = Hscv.diag(x = data)

These bandwidths can be used to compute a kernel density estimate using kde command:

kde(x=X, H=bw1)

Other important arguments of kdeare gridsize, eval.pointsand binned. Gridsize controls the number of grid points; kde compute density at specific eval.points if supplied or estimate density over a grid defined by gridsize. The default value of binned is FALSE; binned estimation is used if it is TRUE.

Monte-Carlo simulation for bandwidth selection

We conduct a simulation study to select best bandwidth for estimating bivariate skew-normal and skew-t densities.

To select the best bandwidth for estimating bivariate densities, we run these methods on simulated data and compare the distances between estimated and true densities. We are particularly interested in skewed density since there are empirical evidences that most asset distributions are asymmetric (Xiong et al., 2011; Bonato, 2011). Our target densities are skew-normal and skew-t densities (Azzalini and Valle, 1996) with different shape parameters: (i) skew-normal with shape$(0.5, 0.4)$, (ii) skew-normal with shape $(-0.5, -0.4)$, (iii) skew-t with shape $(0.5, 0.4)$ and (iv) skew-t with shape $(-0.5, -0.4)$. We generate data from these distributions with sample sizes, $n = 1000$ and $n = 100$. We then estimate densities with all these bandwidth selectors and calculate the MISE in each case. The experiment is replicated 1000 times. Figure 1 display boxplots of log of MISEs for different bandwidth selectors and target densities.

It is desired that MISEs are close to zero or equivalently log of MISEs are large negative number. We notice that both full matrix and diagonal cross validation bandwidth selector given by Rudemo (1982) and Bowman (1984) has larger dispersion than other bandwidth selectors considered in this study. The biased cross validation (BCV) bandwidth of Sain et al. (1994), smooth cross validation (SCV) bandwidth of Duong and Hazelton (2005) and all the plug-in bandwidth selectors give consistent results regardless of target densities. Among these eight methods of bandwidth selection, cross validation bandwidth selectors have lower MISE compared to plug-in counterparts. Further, we observe that Duong and Hazelton's (2005) full matrix bandwidth selector has little lower discrepancy (inter-quartile range) than others.

3.3.1.3 Comparison between histogram and kernel density

Kernel can be superior to the histogram in terms of (i) a better mean squarer error rate of convergence of the estimate to the underlying density, (ii) an insensitivity to the choice of origin, and (iii) the ability to specify more sophisticated window shapes than the rectangular window for binning or multivariate counting (Silverman, 1986; Devroye, 1987).

While the histogram is easy to comprehend, it has several drawbacks. It is discontinuous and changes with the choice of the origin and bin width. Silverman illustrates these problems graphically. Histogram construction is such a routine process that may fail to realize that even when using identical bin widths, different origin choices may change the histogram significantly. Clearly it may be desirable to choose band width h differently by coordinate in the multivariate setting.

Suppose we have a random sample $X_1,\dots,X_n$ taken from a continuous, univariate density f. Suppose the knots are $x_0,\dots,x_n$ where $x_k = x_0 + kb$. Since f is a density function, denote the c.d.f.

$$F(x) = \int_{-\infty}^{x} f(x)dx.$$

The histogram could be written as

$$\hat{f}(x;b) = \frac{1}{nb}\sum_{i=1}^{n} I_{(x_k,x_k+b)}(X_i) \text{ where } X\epsilon\,(x_k, x_k+b]$$

Then

$$E\hat{f}(x;b) = \frac{1}{b}\int_{x_k}^{x_k+b} f(x)dx = \frac{F(x_k+b)-F(x_k)}{b} = f(x_k) + \frac{b}{2}f'(x_k) + o(b)\ ;$$

$$Bias = f(x_k) - \{f(x_k) + (x-x_k)f'(x_k) + o(x-x_k)\} + \frac{b}{2}f'(x_k) + o(b)$$

$$= \left\{\frac{b}{2} - (x-x_k)\right\}f'(x_k) + o(b)\ ;$$

$$E\hat{f}^2(x;b) = \frac{1}{nb^2}\{F(x_k+b) - F(x_k)\} + \frac{n(n-1)}{n^2b^2}\{F(x_k+b) - F(x_k)\}^2;$$

$$V\hat{f}(x;b) = \frac{1}{nb}\{f(x_k) + o(1)\} - \frac{1}{n}\{f(x_k) + o(1)\}^2.$$

Vary x in different bins, and take integration, we have

$$\text{MISE}\ \{\hat{f}(.;h)\} = AMISE\{\hat{f}(.;h)\} + o\{(nb)^{-1} + b^2\};$$

$$\text{AMISE}\ \{\hat{f}(.;h)\} = \frac{1}{nb} + \frac{b^2}{12}R(f')$$

Therefore, MISE is asymptotically minimized at

$$b_{MISE} \sim \left(\frac{6}{R(f')}\right)^{\frac{1}{3}} n^{-\frac{1}{3}} \ , \ \inf MISE\{\hat{f}(x;h)\} \sim \frac{1}{4}\{36R(f')\}^{\frac{1}{3}} n^{-\frac{2}{3}} .$$

Thus, MISE of histogram has a convergence rate of $o(n^{\frac{-2}{3}})$

Now, a kernel density is defined as

$$\hat{f}(x;h) = \frac{1}{nh}\sum_{i=1}^{n} K\left(\frac{x - X_i}{h}\right)$$

$$E\hat{f}(x;h) = EK_h(x - X) = (K_h * f)(x)$$

$$= \int \frac{1}{h} K\left(\frac{x-y}{h}\right) f(y)dy = \int K(z)f(x - hz)dz.$$

Expand $f(x - hz)$ about x, we obtain that:

$$f(x - hz) = f(x) - hf'(x) + \frac{1}{2}h^2 z^2 f''(x) + o(h^2)$$

which is uniformly in z, hence

$$E\hat{f}(x;h) - f(x) = \frac{1}{2}h^2 \mu_2(k) f''(x) + \ o(h^2)$$

Similarly,

$$V\hat{f}(x;h) = n^{-1}\{(K_h^2 * f)(x) - (K_h * f)^2(x)\}$$

$$= \frac{1}{nh}\int K^2\,(z)f(x - hz)dz - n^{-1}\int K(z)\{f(x - hz)dz\}$$

$$= \frac{1}{nh}\int K^2\,(z)\{f(x) + o(1)\}dz - n^{-1}\int K(z)\{f(x) + o(1)\}dz$$

$$= \frac{1}{nh} R(K)\{f(x) + o(\tfrac{1}{nh})\}.$$

Therefore,

$$MSE\{\hat{f}(x;h)\} = \frac{1}{nh}R(K)\{f(x) + \frac{1}{4}h^4\mu_2^2(K)f''^2(z) + o\{\frac{1}{nh} + h^4\};$$

$$MISE\{\hat{f}(.;h)\} = AMISE\{\hat{f}(.;h)\} + o\{\frac{1}{nh} + h^4\};$$

$$AMISE\{\hat{f}(.;h)\} = \frac{1}{nh}R(K) + \frac{1}{4}h^4\mu_2^2(K)R(f'').$$

Notice that the tail term $o\{\frac{1}{nh} + h^4\}$ shows the variance-bias trade-off, while $AMISE$ could minimized at

$$h_{AMISE} = [\frac{R(K)}{n\mu_2^2(K)R(f'')}]^{\frac{1}{5}} \ , \ \inf AMISE\{\hat{f}(x;h)\} = \frac{5}{4}\{\mu_2^2(K)R(K)R(f'')\}^{\frac{1}{5}}n^{\frac{-4}{5}}$$

Equivalently, as $n \to \infty$, we can rewrite

$$h_{MISE} = [\frac{R(K)}{n\mu_2^2(K)R(f'')}]^{\frac{1}{5}} \ , \ \inf MISE\{\hat{f}(x;h)\} \simeq \frac{5}{4}\{\mu_2^2(K)R(K)^4R(f'')\}^{\frac{1}{5}}n^{\frac{-4}{5}}$$

Aside from its dependence on the known K and n, the expression shows us the optimal h is inversely proportional to the curvature of f, i.e.$R(f'')$. The best obtainable rate of convergence of the $MISE$ of the kernel estimator is of order$n^{\frac{-4}{5}}$.

We thus conclude that, the $MISE$ of histogram is asymptotically inferior to the kernel estimator, since its convergence rate is $o(n^{\frac{-2}{3}})$ compared to the kernel estimator's $o(n^{\frac{-4}{5}})$ rate.

We also compare the histogram and kernel density with a simulation experiment. Suppose X and Y follows a bivariate normal distribution with mean $\mu_x = \mu_y = 0$ and variance covariance matrix $\Sigma = \begin{bmatrix} 4 & 2 \\ 2 & 3 \end{bmatrix}$. We simulate observations of different sample

size $(n = 500, 1000, 10000)$ and estimate univariate and joint entropy of these simulated data with both histogram and kernel density. The procedure is replicated $10{,}000$ times. Table3.1 displays MSEs of entropy estimation. We observe that in case of large sample $(n = 10000)$ MSEs in entropy estimation for histogram is equal to that for kernel density refers that for large sample, histogram and kernel density are equivalent. However, for small to moderate sample$(n = 500, 1000)$, kernel density provides smaller MSE for entropy estimation than the histogram. Thus, we may conclude that kernel density has a better small sample property than histogram.

Table 3.1: MSE in Entropy Estimation

n	**Hx**		**Hy**		**Hxy**	
	Histogram	**Kernel**	**Histogram**	**Kernel**	**Histogram**	**Kernel**
500	0.32	0.02	0.18	0.08	0.95	0.21
1000	0.22	0.01	0.11	0.05	0.62	0.12
10000	0.01	0.01	0.01	0.01	0.01	0.01

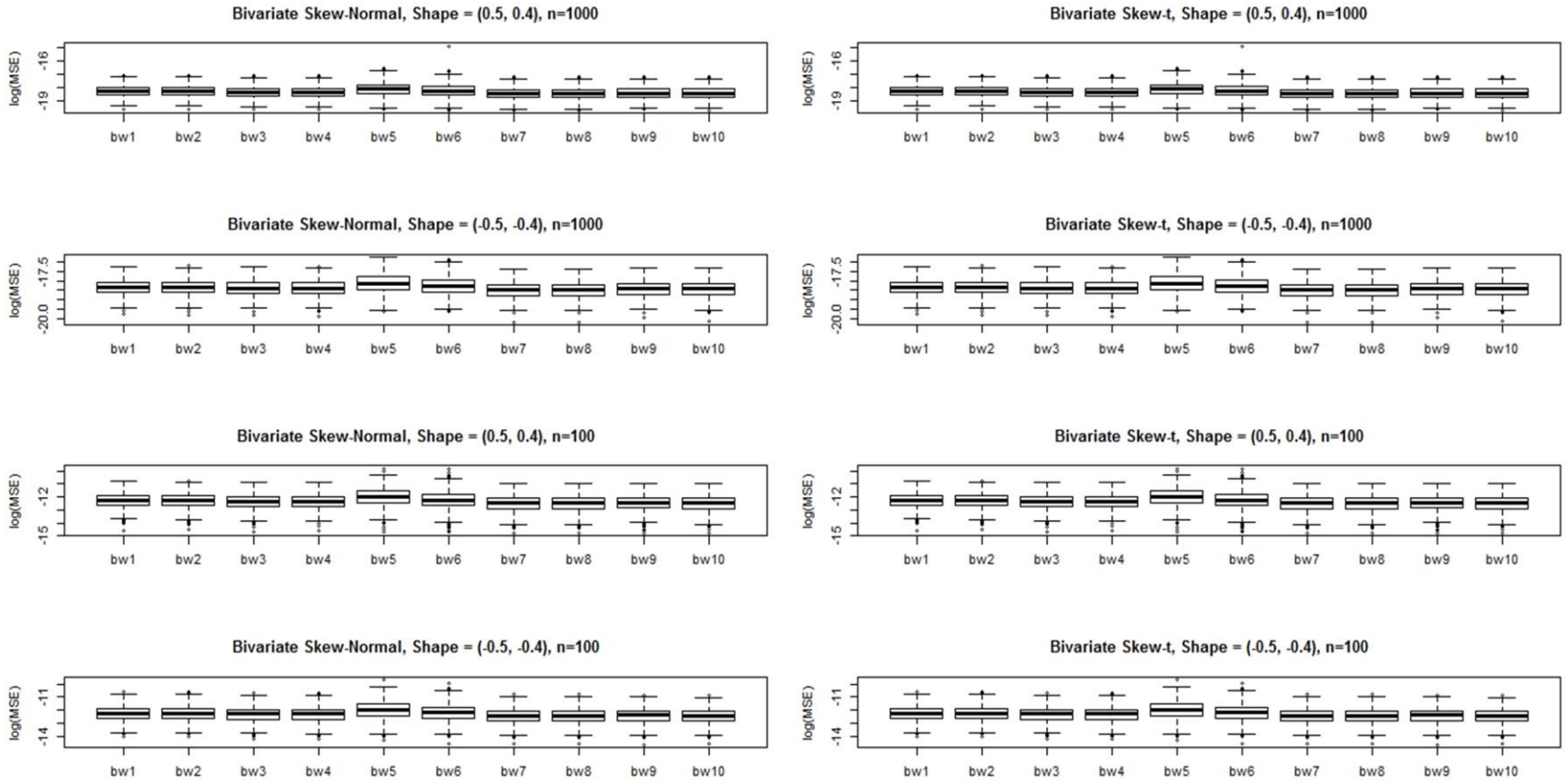


Figure 3.1: Spread of different bandwidth selectors

3.4 Scale of measurement

For a known distribution, a relationship between entropy and variance can be established. If entropy exists, it is usually a function of the variance. For example, if σ^2 is the variance of a normal variate X, the corresponding entropy is $\frac{1}{2}(1+\log 2\pi\sigma^2)$. Thus, as a measure of dispersion they seem to be equivalent. However, since the scale of measurement for variance and entropy are different, they are not comparable directly in analyzing real data. Figure 3.2 displays the entropy of different distributions against the corresponding standard deviation. As we observe entropy is increasing as the standard deviation (variance) is increasing. However, the rate of change in entropy decreases gradually as the standard deviation (variance) increases. This feature makes the basic difference between practical use of entropy and variance. For example, let, X_1 and X_2 be two normally distributed assets with corresponding variance of 2 and 40, and corresponding entropy 2.112 and 3.263. The comparative riskiness between these two assets may be interpreted differently by entropy than that by variance. We propose a risk measure based on entropy which is equivalent to standard deviation (variance) in terms of scale measurement. Table 3.2 provides variance, entropy (H) and $exp(H)$ for different distributions. For all these distributions, we notice that H is a nonlinear function of standard deviation (variance). But, $exp(H)$ is a linear function of standard deviation. For a normal distribution, for example, we have

$$\exp(H) = \left((2\pi)^{\frac{1}{2}} exp(1/2)\right) \times \sigma = k \times \sigma$$

So that

$$\exp(H) \propto Standard\ Deviation$$

We, therefore, suggest using $exp(H)$ as a nonparametric and distribution free measure of dispersion and risk which is equivalent to standard deviation in terms of scale of measurement.

Table 3.2: Entropies of some probability distributions

Distribution	***pdf***	**Variance**	**Entropy (H)**	***exp*(H)**
Normal	$\frac{1}{\sqrt{2\,\pi\sigma^2}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$	σ^2	$\frac{1}{2}(1+\log(2\pi\sigma^2))$	$\left((2\pi)^{\frac{1}{2}} exp(1/2)\right)\times\sigma$
Exponential	$\frac{1}{\lambda} e^{\frac{-x}{\lambda}}$	λ^2	1+logλ	$(exp(1))\times\lambda$
Rayleigh	$\frac{x}{\sigma^2} e^{\frac{-x^2}{2\sigma^2}}$	$\left(\frac{4-\pi}{2}\right)\sigma^2$	$1+\ln\frac{\sigma}{\sqrt{2}}+\frac{\gamma}{2}$	$\left((4-\pi)^{\frac{1}{2}} exp(1+\frac{\gamma}{2})\right)\times\left(\frac{4-\pi}{2}\right)^{\frac{1}{2}}\sigma$
Logistic	$\frac{e^{\frac{-(x-\mu)}{\sigma}}}{\sigma(1+e^{\frac{-(x-\mu)}{\sigma}})^2}$	$\frac{\pi^2}{3}\sigma^2$	log(σ)+2	$\left(\frac{3^{\frac{1}{2}}\exp(2)}{\pi}\right)\times\frac{\pi}{3^{\frac{1}{2}}}\sigma$
Uniform	$\frac{1}{\beta-\alpha}$	$\frac{(\beta-\alpha)^2}{12}$	$\log(\beta-\alpha)$	$12^{\frac{1}{2}}\times\frac{\beta-\alpha}{12^{\frac{1}{2}}}$
Binomial	$\binom{n}{x} p^x(1-p)^{n-x}$	$np(1-p)$	$\frac{1}{2}\log(2\pi np(1-p))+O\left(\frac{1}{n}\right)$	$\left((2\pi)^{\frac{1}{2}}\exp\left(O\left(\frac{1}{n}\right)\right)\right)\times(np(1-p))^{\frac{1}{2}}$

Table 3.2: Entropies of some probability distributions (continued)

Laplace	$\frac{1}{2\sigma}e^{-\frac{\lvert x-\alpha\rvert}{\sigma}}$	$2\sigma^2$	$1+log(2\sigma)$	$\left(2^{\frac{1}{2}}\exp(1)\right)\times 2^{\frac{1}{2}}\sigma$
Cauchi	$1/(\pi\lambda(1+\frac{(x-x_0)^2}{\lambda}$	undefined	log(λ)+log(4π)	$4\pi\times\lambda$
Erlang	$\frac{\lambda^k}{(k-1)!}x^{k-1}e^{(-\lambda x)}$	$\frac{k}{\lambda^2}$	$(1-k)\psi(k)+\ln\left(\frac{\Gamma(k)}{\lambda}\right)+k$	$\left((\Gamma k/\sqrt{k})exp\big((1-k)\psi(k)+k\big)\right)\times\frac{\sqrt{k}}{\lambda}$
Gamma	$\frac{\beta^\alpha}{\lrcorner(\alpha)}x^{\alpha-1}e^{-\beta x}$	$\frac{\alpha}{\beta^2}$	α+log$\beta+log_{\lrcorner}(\alpha)+(1-\beta)_{\lrcorner}(\alpha)$	$\left(\frac{\Gamma\alpha}{\sqrt{\alpha}}exp(\alpha+(1-\alpha)\psi(\alpha))\right)\times\frac{\sqrt{\alpha}}{\beta}$
Multivariate Normal	$(2\pi)^{-\frac{k}{2}}\lvert\Sigma\rvert^{-\frac{1}{2}}e^{-\frac{1}{2}(x-\mu)'\Sigma^{-1}\{x-\mu\}}$	Σ	$\frac{k}{2}(1+\ln(2\pi))+\frac{1}{2}\ln\lvert\Sigma\rvert$	$\left(2\pi\,exp\left[\frac{k}{2}\right]\right)\times\Sigma^{\frac{1}{2}}$

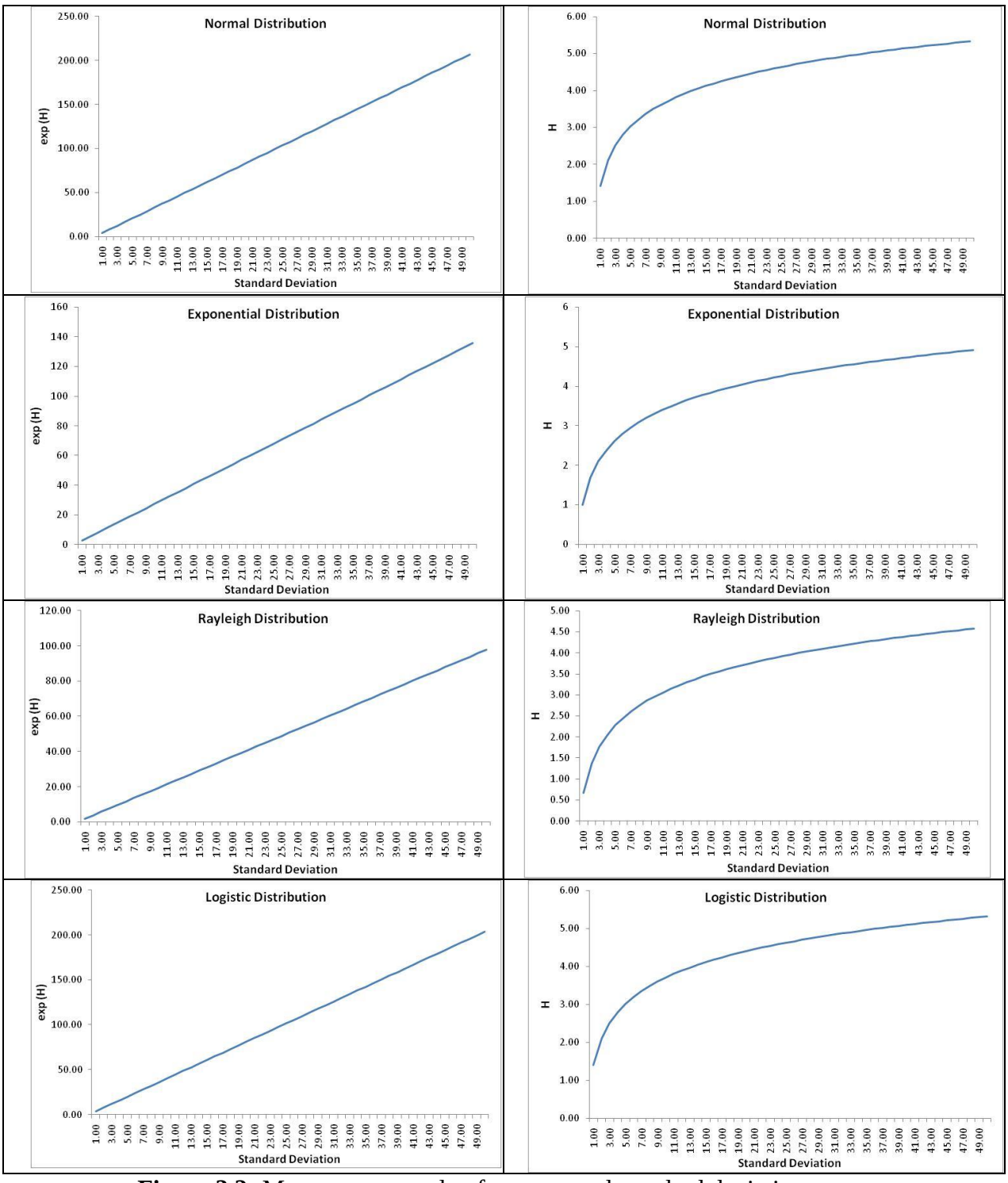


Figure 3.2: Measurement scale of entropy and standard deviation

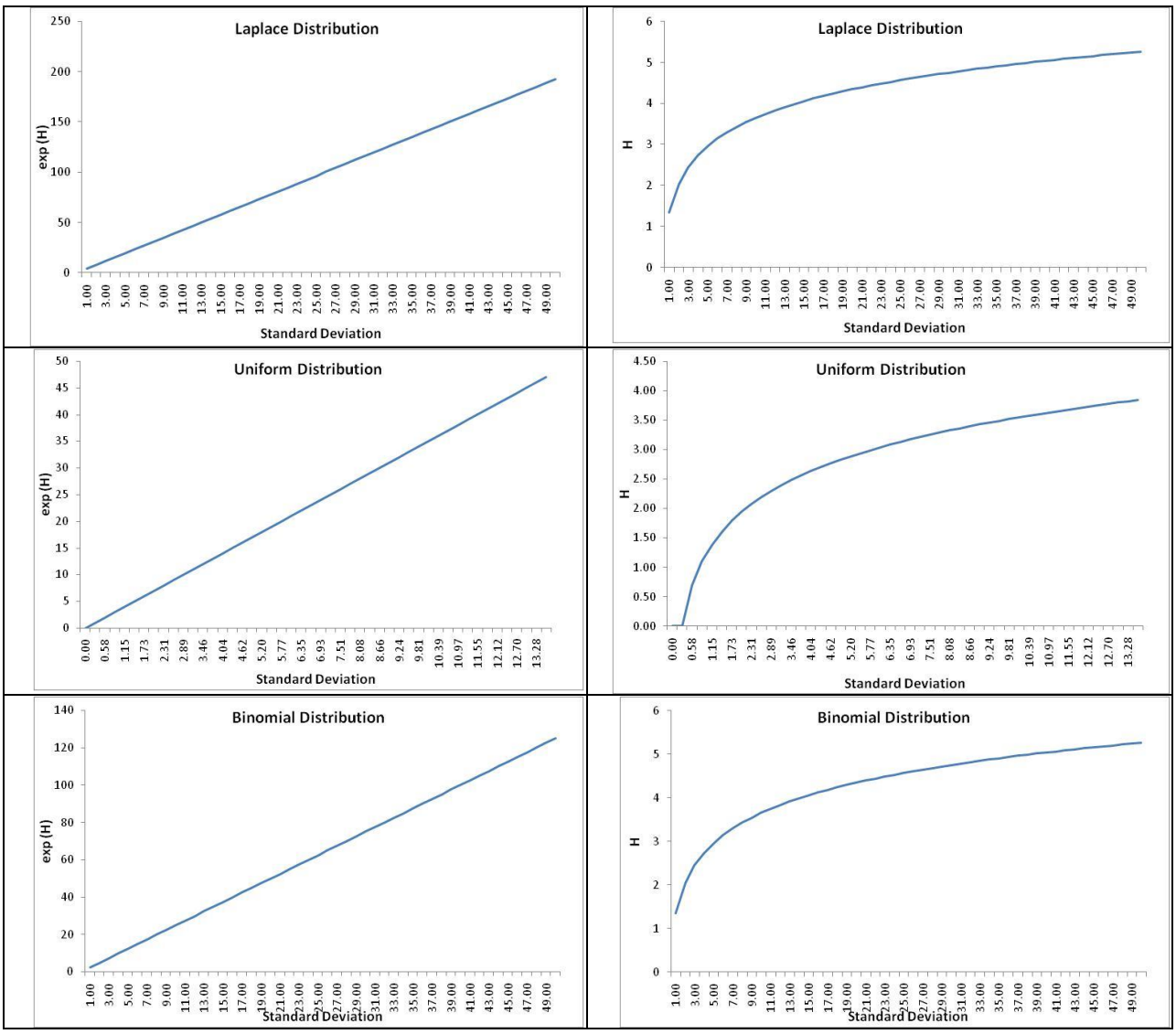


Figure 3.2 (cont.): Measurement scale of entropy and standard deviation

3.5 Diversity

Markowitz (1952) formally developed the concept of portfolio diversification and show that diversification leads to risk reduction. However, "perfect diversification" does not mean the absence of risk, nor does it mean an optimally balanced portfolio. In the Markowitz model the variability of expected portfolio returns is reduced as the number of assets increases and the specific risk within the portfolio is reduced (see Figure. 3.3). The total risk can be divided into two parts: unsystematic risk or specific risk and systematic risk or market risk. Unsystematic risk is also called diversifiable risk. That is the portion of the total risk that is peculiar or unique to a firm. Systematic risk is that portion of total risk caused by factors affecting all the economy. In other words, portfolio risk can be decomposed as systematic risk and unsystematic risk or specific risk, which is contributed by the individual assets and can be reduced in a well-diversified portfolio. The motivation of controlling the specific risk is that the risk coming from specific sources of the individual assets is more volatile and uncertain.

It is interesting to note that the total risk of an asset can be split into systematic risk and specific risk with the notion of entropy. Let X be an asset and Y represent a market index. Then the total risk, $H(X)$ in the X can be decomposed as

$$H(X) = I(X,Y) + H(X|Y),$$

where $I(X,Y)$ is the mutual information of the asset X and the market index Y that measures the share of risk common in the market and, therefore, can be treated as systematic risk. The, $H(X|Y)$ is the conditional entropy of X given the market index Y that measures the risk of X after separating the market risk and, therefore, can be treated as the specific risk.

In this section, we examine whether entropy respond to diversification and if so, how sensitive the entropy is due to diversification. Being a convex function, the standard deviation (variance) is a well-known and popular measure of risk that is sensitive to diversification. At the same time, the entropy (H) is a concave function and negative of entropy ($-H$) is a convex. Moreover, the subadditivity property $H(X,Y) \leq H(X) + H(Y)$[1] ensures that this risk measure is sensitive to diversification. Rao (1984) discuss the conditions and desirable properties of different entropy based diversity measures and noted that use of entropy as a measure of diversity has some advantages since is also applicable to non-metric data.

Elton and Gruber (1995) empirically verified how diversification can be a factor of risk reduction, where they use variance as a measure of risk. In their experiment they use equally weighted portfolio model for randomly selected assets. They come out with a conclusion that as the number of assets increases, the portfolio risk measured by variance (standard deviation) decreases. Here, we conduct an experiment somewhat similar to Elton and Gruber (1995) and Dionisio et al. (2005) to verify empirically how entropy respond to the effect of diversity. We use monthly closing prices of 15 stocks rated on the New York stock exchange (*NYSE*), spanning from Aug 2004 to June 2013, which corresponds to 107 observations *per* stock. Unlike the previous literature, we first ordered the assets in terms of the magnitude of risk; starting with the most risky asset we add all the assets, in the portfolio sequentially. The specific risks of assets are measured with the conditional entropy of assets for given the market index. For the purpose of comparison, standard deviations of the portfolios are replaced by the normal entropies since standard deviation is not directly comparable with entropy in terms of

[1] For proof see Appendix II

metric of measurement while the normal entropy is equivalent to standard deviation (see Table 3.2 for detail).

Our results (see Figure 3.4) show that both the empirical entropy and the standard-deviation (normal entropy) of the portfolios decreases gradually as more assets are added. These results can be explained by the fact that risk/uncertainty of the portfolio decreases with increasing number of assets. In other words, diversification makes a portfolio less risky which can be measured in terms of standard deviation or entropy. Thus, it can be inferred that like standard deviation entropy is also sensitive to the effect of diversification.

This experiment also provides an empirical evidence of the subadditivity rule for entropy:

$$H[\theta X] + H\,[(1-\theta)\,Y] \geq H[\theta X + (1-\theta)Y],$$

where $\theta = 1/N$,

which is a desired property of a portfolio risk measure (Reesor and McLeish, 2002). We further note that, in this experiment, the normal entropy is always greater than the empirical entropy. We can therefore, infer that the predictability level of portfolio maybe underestimated (or risk maybe overestimated) when assets are assumed to be normally distributed. From this analysis, we can conclude that entropy is a distribution free and more informative risk measure than the variance; besides, it can capture the effect of diversification.

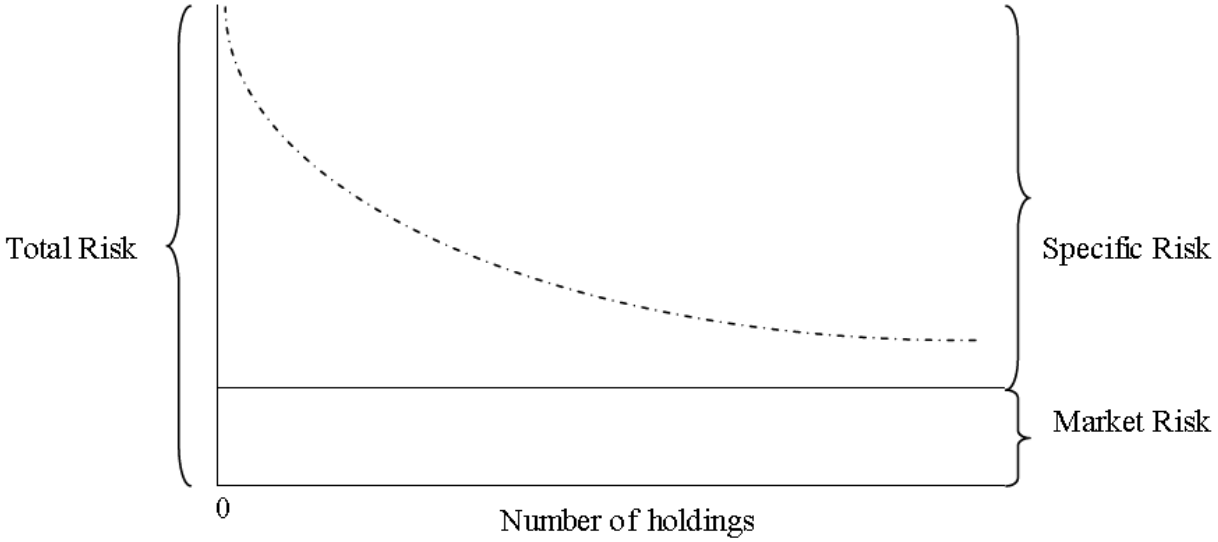


Fig. 3.3: Risk reduction by diversification

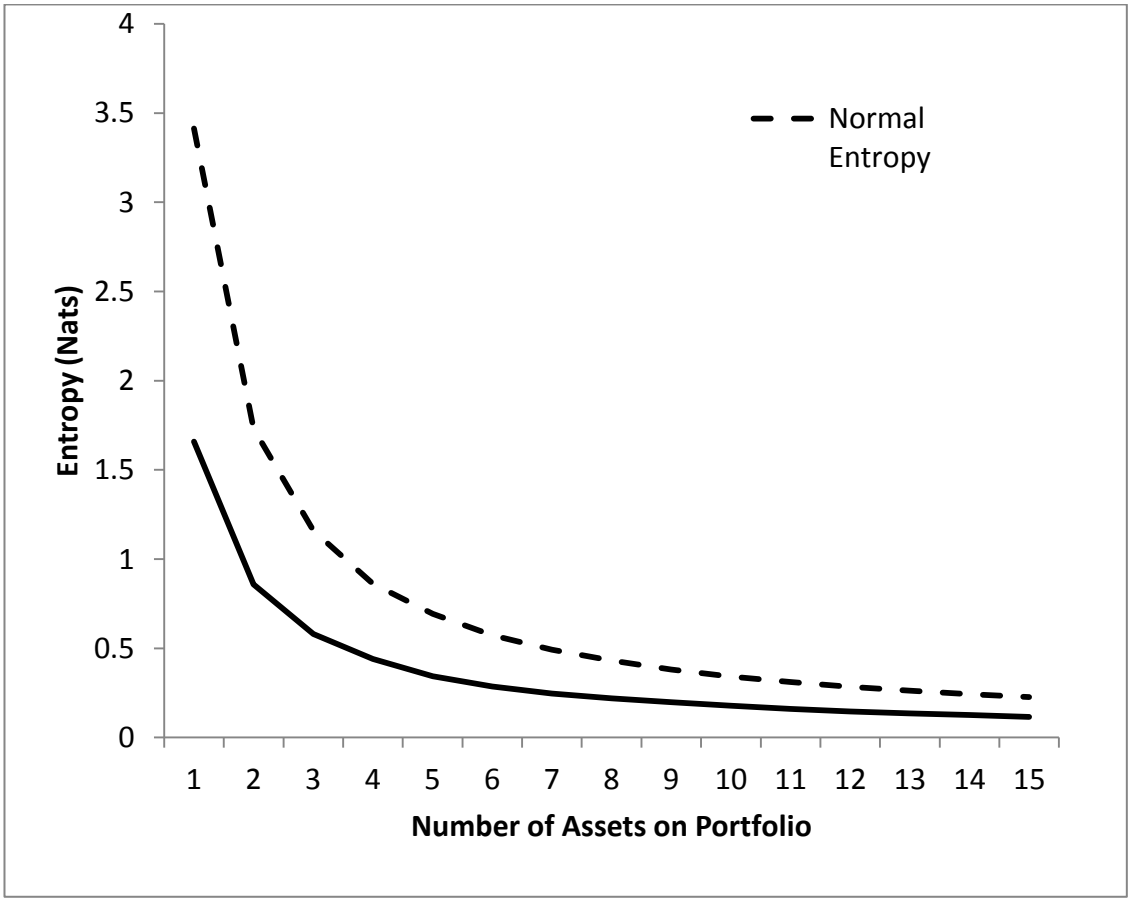


Fig. 3.4: Comparative analysis of the empirical entropy (*H*) and the normal entropy (*NH*) for portfolios randomly selected. Entropy is measured in *nats* because we use natural logarithms.

CHAPTER 4: MULTI-OBJECTIVE PORTFOLIO MODELS

4.1. Introduction

Portfolio optimization has been the object of intense research and is still developing. Markowitz's (Markowitz, 1952) mean-variance (MV) efficient portfolio selection is one of the most widely used approaches for asset allocation and is very popular among practitioners. However, some drawbacks of this approach are pointed out in the literature. Bera and Park (2008) argue that MV approach, based on sample moments like mean and variance often concentrates on few assets only and thus leads to less diversified portfolio. Lack of attention to the uncertainty in the data and adoption of wrong model, sample estimates of mean and variance can be poorly estimated (Jobson and Korkie, 1980) and hence portfolio optimization based on inaccurate point estimates may be highly misleading. In some cases, variations in the input data may greatly affect the portfolio greatly and even a few new observations may change the portfolio completely. Demiguel (2009) noted that the out-of-sample performance of MV portfolio is sometimes, no better than the naive $\frac{1}{N}$ benchmark. In addition, empirical evidences show that almost all asset classes and portfolios have returns that are not normally distributed (Xiong et al., 2011), and the first and second moments are generally insufficient to explain portfolios in the case of non-normal return distribution (Usta and Yeliz, 2010). Ke and Zhang (2008) noted another limitation of MV model, that is, the standard deviation cannot perfectly represent the risk, because the sign of error does not affect the fluctuation. However, many assets' return distributions are asymmetrical; also, most asset return distributions are more leptokurtic, or fatter tailed, than are normal distribution. Patton (2004) showed that knowledge of both skewness and asymmetric dependence leads to economically significant gains. Recent research (Müller, 2010, for example) suggests that higher moments are important considerations in asset allocation.

Investors are particularly concerned about significant losses, that is, the downside risk, which is a function of skewness and kurtosis.

The MV approach is a single-objective optimized portfolio that cannot satisfy all investors demands or constraints. Thus, the need to accommodate multiple criteria renders the hypothesis of a single-objective function to be optimized subject to a set of constraints is no longer suitable, and the introduction of a multi-objective optimization framework allows one to manage more information. For instance, both risk minimization and diversification can be achieved through a bi-objective portfolio optimization. Multi-objective portfolio models with Fuzzy programming are discussed in Samanta and Roy (2005) and Jana et al. (2007; 2009). Usta and Kantar (2011) show that the empirical performance of a multi-objective model is better than that of a single-objective model. The portfolio model by Ke and Zhang (2008) has two objectives: variance minimization and entropy maximization. Shirazi et al. (2013) argue that in multi-objective models, the use of portfolio entropy instead of portfolio variance ensures proper estimate of risk in case of non-normal asset distribution since entropy depends on higher order moments than variance and it is not restricted to a particular distribution. However, their argument is based on an empirical evaluation of one small set of equity market data.

Most of the investors have multiple investment objectives and the traditional single-objective mean variance optimization approach is not adequate to meet their demands. Thus, the increase in application of multi-objective optimization in portfolio selection problem is magnified. In this chapter, we propose a multi-objective portfolio model based on entropy which ensures a proper use of historical risk and is well diversified. The model is formulated in a generalized form that represents a class of portfolio models, where a single-objective model is a special case. We compare the performance

of the multi-objective portfolios with that of single-objectives in presence of stock markets. A rolling window procedure is used to compare empirical performances of single-and multi-objective portfolio models.

4.2. Multi-Objective Optimization

Mathematically multi-objective optimization can be expressed as:

$$\min[f_1(x), f_2(x), \ldots, f_n(x)], \qquad x \epsilon S, \quad n > 1, \tag{4.1}$$

where S is the set of the constraint. As a special case, a bi-objective optimization can be expressed as:

$$\min\ [f_1(x), f_2(x)], \quad x \epsilon S \tag{4.2}$$

Let the objective space C be the space in which objective vector belongs. Such a set is defined as:

$$C = \{y \epsilon R^n : y = f(x), x \epsilon S\} \tag{4.3}$$

In a multi-objective setting, we need the concept of Pareto optimality. A vector $x^* \epsilon S$ is said to be Pareto optimal for multi-objective problem if all other vector $x \epsilon S$ have a higher value for at least one of the objective function $f_j (j = 1,2, \ldots n)$ or have the same value for all the objective functions. This can be defined as:

A point $x^*$is said to be a weak Pareto optimum or a weak efficient solution for the multi-objective problem if and only if there is no $x \in S$ such that $f_i(x) < f_i(x^*)$ for all $i \in \{1,2, \ldots, n\}$.

A point x^* is said to be a strict Pareto optimum or a strict efficient solution for the multi-objective problem if and only if there is no $x \in S$ such that $f_i(x) \leq f_i(x^*)$ for all $i \in \{1,2, \dots, n\}$, with at least one strict inequality.

We can also speak of locally Pareto-optimal points, for which the definition is the same as above, except that we restrict attention to a feasible neighborhood of x^*. In other words, if $B(x^*, \varepsilon)$ is a ball of radius ε >0 around point x^*, we require that for some $\varepsilon > 0$, there is no $x \in S \cap B(x^*, \varepsilon)$ such that $f_i(x) \leq f_i(x^*)$ for all $i \in \{1,2, \dots, n\}$, with at least one strict inequality.

The image of the efficient set, that is the image of all the efficient solutions, is called Pareto front or Pareto curve or surface. The shape of the Pareto surface indicates the nature of the trade-off between the different objective functions. An example of a Pareto curve is present Figure 4.1, where all the points between $\left(f_2(\hat{x}), f_1(\hat{x})\right)$ and $\left(f_2(\tilde{x}), f_1(\tilde{x})\right)$ define the Pareto front. These points are called non-inferior or nondominated points.

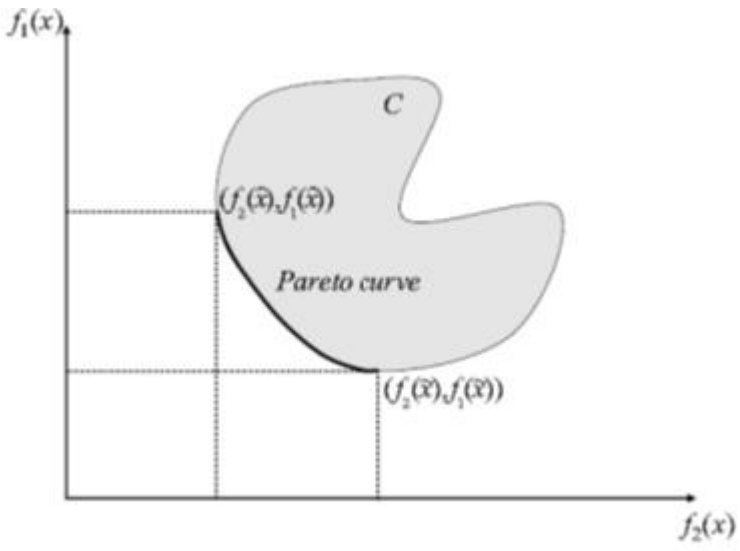


Figure 4.1: Example of Pareto curve

4.3. Entropy based Multi-Objective Portfolio Model

A diversified portfolio model should have at least two objectives: risk minimization and diversification, which can be formulated as

$$\begin{Bmatrix} \min \sum_{i=1}^{n} x_i{}^2 C_i \\ \max(-\sum_{i=1}^{n} x_i ln x_i) \end{Bmatrix}$$

such that

$$\sum_{i=1}^{n} x_i R_i = d_0 \ ,$$

$$\sum_{i=1}^{n} x_i = 1 \qquad (4.4)$$

where C_i= Risk associated with the i_{th} asset.

R_i =Return from the i_{th} asset.

d_0 =Expected portfolio return

x_i=Portfolio weight for the i_{th} asset.

This problem is equivalent to:

$$\min f_1(x) = \min \sum_{i=1}^{n} x_i^2 \, C_i \quad ,$$

$$\max f_2(x) = \max \sum_{i=1}^{n} x_i \ln x_i \qquad (4.5)$$

such that

$$\sum_{i=1}^{n} x_i R_i = d_0 \ ,$$

$$\sum_{i=1}^{n} x_i = 1$$

4.3.1 Solution to the multi-objective optimization

A Pareto solution for multi objective optimization may not be straightforward. Approximations can help in such cases. One approach to solve this kind of problems is scalarization. This involves combining multiple objectives into one single objective scalar function:

$$\min \sum_{j=1}^{n} \gamma_j f_j(x), \tag{4.6}$$

$$\sum_{j=1}^{n} \gamma_j = 1, \ \gamma_j > 0, \ j = 1,2, \ldots . n \quad , \quad x \epsilon S.$$

The bi-objective portfolio model in eq. (4.5) can be written in the following form

$$\min(\gamma_1 \sum_{i=1}^{m} {x_i}^2 C_i + \gamma_2 \sum_{i=1}^{m} x_i ln x_i) \tag{4.7}$$

such that $\sum_{i=1}^{m} x_i R_i = d_0$ and $\sum_{i=1}^{m} x_i = 1.$

Equivalently we can write

$$\min(\sum_{i=1}^{m} {x_i}^2 C_i + \xi \sum_{i=1}^{m} x_i ln x_i) \tag{4.8}$$

such that $\sum_{i=1}^{m} x_i R_i = d_0$ and $\sum_{i=1}^{m} x_i = 1$ and $\xi = \frac{\gamma_2}{\gamma_1}$

Equation (4.8) is a class of portfolio models. In this model, ξ is called the momentum factor that determines the trade-off between historical risk and diversity. If the future risk is different from the historical risk, this multi-objective model certainly performs better than the single-objective model. If $\xi = 0$, and the risk, C_i is replaced by corresponding variance estimate, this model is single-objective and is called mean-variance (MV) model; if C_i is replaced by corresponding entropy estimate, this model is

called mean-entropy (ME) model. In cases when $\xi > 0$ and the risk C_i is replaced by corresponding variance estimate, the model is called mean-variance-entropy (MVE) model; if C_i is replaced by corresponding entropy estimate, this model is called mean-entropy-entropy (MEE) model. The later has some interesting features: entropy is used for measuring both risk and diversity. Use of entropy as an alternative measure of risk especially for non-normal data has been suggested in the literature (Philippatos and Gressis, 1975; Philippatos and Wilson, 1972; Nawrocki and Harding, 1986). Estimation of entropy risk for an asset given the market index is discussed in Shirazi et al (2013).

Apparently, the Lagrange multiplier technique can be used to solve eq. (4.8). Since the objective functions are nonlinear, an exact expression for *x* is not available. However, a number of methods such as the argumented Lagrange multiplier with sequential quadratic programming(SQP) interior algorithm are available to solve this type of nonlinear problem.

4.4. Illustration

Let W be the size of window and k be the number of observations to be dropped as we move from one window to the next. If L be the total number of observations, the total number of window will be $(L - W)/k + 1$.We first estimate portfolio weight vector, w_1 , for the sample period $S1$: $(1, W)$ and calculate the return, $R_{p,1}$ and risk, $V_{p,1}$. We then repeat the procedure for the next sample period $S2$: $(1 + k, W + k)$. We proceed this way until the data is exhausted. At the end of the procedure, we will have $(L - W)/k + 1$ portfolio weight vectors. The average of in-sample estimate of the Sharpe Ratio (SR) is calculated as

$$SR_{in} = \frac{1}{((L-W)/k+1)} \sum_{t} \frac{w_t' R_{p,t}}{\sqrt{w_t' V_{p,t} w_t}}$$

In the same way, we calculate other performance measures alternative to SR (for example, Bera and park, 2008; Usta and Kantar, 2011): certainty equivalent return (CEQ), adjusted Sharpe ratio (ASR), mean absolute deviation ratio ($MADR$), Sortino-Satchell ratio (SSR) and Farinelli-Ferreira-Rossello Ratio (FTR).

The portfolio turnover (PT) is defined as the average absolute change in the weights and its formula is given as follows

$$PT = \frac{1}{(L-W)/k} \sum_t \sum_{i=1}^{n} \left| w_{i,t+1} - w_{i,t} \right|,$$

where $w_{i,t}$ is the *i*-th portfolio weight for the *t*-th window.

The out-of-sample return of portfolio in period $t+1$, denoted by $R_{p,t+1}$, is calculated by $R_{p,t+1} = w_t' R_{t+1}$, where $R_{t+1} = \left(R_{1,t+1}, R_{2,t+1}, \ldots R_{n,t+1}\right)$denotes the return vector in period $t+1$. Similarly, the portfolio variance is calculated as $V_{p,t+1} = w_t' V_{t+1} w_t$.

The average of out-of-sample estimate of the SR is calculated as

$$SR_{out} = \frac{1}{(L-W)/k} \sum_{t} \frac{w_t' R_{p,t+1}}{\sqrt{w_t' V_{t+1} w_t}}$$

To measure the diversity of an estimated portfolio, we use two diversity indices proposed by Woerheide (1993).

$$DI_1 = 1 - \sum_{i=1}^{n} w_i^2 \quad \text{and} \quad DI_2 = 1 - w_1 - \sum_{i=2}^{n} w_i^2 [1 + (1 - w_i)]$$

where w_1 is the largest single portfolio weight. A value 0 of the above indices indicates no diversification and a value 1 indicates ultimate diversification.

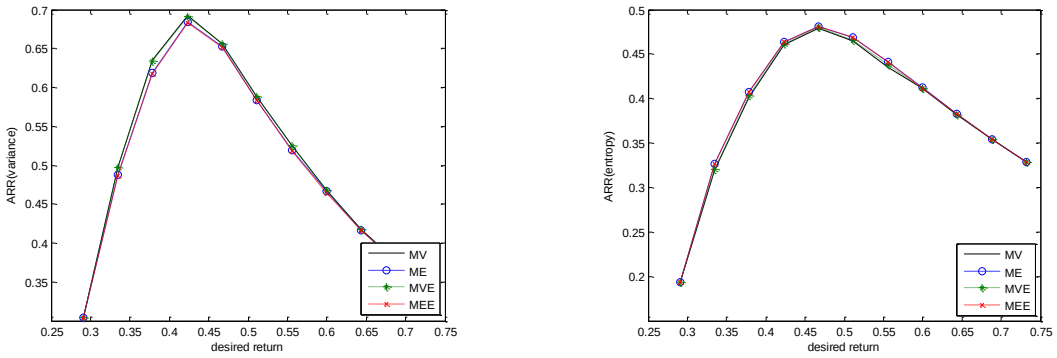
4.4.1 Monte-Carlo Simulation

We apply four portfolio models, discussed above, on a simulated data that consists of four normally distributed variables (say four stocks) correlated with another variable (say index). Portfolio weights and their performance measures are reported in Table 4.1. We observe that MEE offers highest portfolio return, highest SR and is most diversified.

We further calculate the SR of these models for different values of desired portfolio returns (Figure 4.2). As we observe, for normally distributed data all the models perform equivalently.

Table 4.1: Portfolio models and their performance in simulated data

Portfolio Weights						
	Risk Measure		**Portfolio Weights**			
	Variance	Entropy	**ME**	**MEE**	**MV**	**MVE**
Variable1	0.9112	2.2472	0.3678	0.3607	0.4486	0.4281
Variable2	3.5850	4.3378	0.1905	0.194	0.1405	0.1503
Variable3	1.8460	3.0069	0.2749	0.2741	0.2716	0.2750
Variable4	4.5446	4.9547	0.1668	0.1711	0.1393	0.1466
Portfolio Performance						
Portfolio Return			0.4109	0.4131	0.3940	0.3985
Portfolio Variance			0.3673	0.3710	0.3490	0.3500
Portfolio Entropy			0.8265	0.8267	0.8558	0.8437
SR (variance)			0.6780	0.6782	0.6670	0.6736
SR (Entropy)			0.4520	0.4543	0.4259	0.4338
Diversity						
DI_1			0.7250	0.7278	0.6859	0.6970
DI_2			0.3852	0.3880	0.3511	0.3598



(a) (b)

Figure 4.2: Sharp Ratio (SR) of different portfolios for normally distributed data

4.4.2 Application of Stock Market Data

We compare the performance of the multi-objective portfolio models with that of single-objective models in presence of four stock markets: Shanghai stock exchange (SSE), Korea exchange (KRX) and New York stock exchange (NYSE). Our empirical datasets consist of 15 stock prices and the market index of these four stock marketsover the period March 02, 2009 to August 23, 2012 and data have been collected from yahoo finance.

Table 4.2: Summary SSE

Stock Name	Mean	Variance	Entropy	Skewness	Kurtosis	Shapiro-Wilk normality test	p-value
Baotou Tomorrow Technology Co., Ltd	-0.1748	194.6860	1.7917	-0.5538	5.1127	0.9571**	0.0025
Shanghai Pudong Development Bank Co., Ltd	2.0772	216.8743	1.7451	0.3477	5.3232	0.9599**	0.0039
Dongfeng Automobile Co., Ltd	0.0709	159.5026	2.1783	-0.1212	2.5622	0.9867	0.4156
Beijing Capital Co., Ltd	1.3795	255.8832	1.5738	0.8807	7.8807	0.9297**	0.0000
Inner Mongolian Baotou Steel Union Co., Ltd	1.5660	269.6855	1.7729	0.9357	6.1707	0.9492**	0.0007
China Minsheng Banking Corp Ltd	2.5599	184.0062	1.7609	-0.0341	4.8372	0.9614**	0.0051
Shandong Iron & Steel Co Ltd	1.3244	280.5489	1.7886	0.2141	4.4819	0.9593**	0.0036
Huaneng Power International Inc	-0.0177	140.8493	1.8232	-0.1596	4.2671	0.9763	0.0679
Huadian Power International Corporation Limited	-0.1146	208.9837	1.7722	-0.3525	4.7115	0.9674*	0.0140
Guangzhou Baiyun International Airport Company Ltd	-0.0757	107.6410	1.5145	-1.6220	10.2298	0.8875**	0.0000
China Merchants Bank Co., Ltd	1.8141	179.1651	1.7264	0.1730	5.3164	0.9555**	0.0020
Beijing GehuaCatv Network Co., Ltd	0.9360	215.3464	1.6581	1.4257	8.4282	0.9096**	0.0000
Hafei Aviation Industry Co., Ltd	1.3221	238.7129	1.7735	0.4583	5.0509	0.9647**	0.0088
Hua Xia Bank Co., Ltd	0.9815	151.0181	1.7991	-0.0546	4.5909	0.9781	0.0946
China Petroleum & Chemical Corporation	0.5207	157.4674	1.8061	-0.6911	4.9195	0.9552**	0.0018
SSE Composite Index	0.4621	85.5838		-0.5887	3.9850	0.9684*	0.0166

Note: ** and * represent the hypothesis of normality is rejected respectively at 1% and 5% level of significance.

Table 4.3: Summary KRX

Stock Name	Mean	Variance	Entropy	Skewness	Kurtosis	Shapiro-Wilk normality test	p-value
Hankuk Steel Wire Co Ltd	1.4061	385.6402	1.6333	0.5499	6.2177	0.9184**	0.0000
TaihanFiberoptics Co Ltd	0.1265	324.4033	1.6923	1.0514	6.5448	0.9298**	0.0000
Kocom Co Ltd	0.5341	223.9532	1.7166	0.9036	6.4239	0.9240**	0.0000
ATLASBX COMPANY LIMITED	3.5648	301.7039	1.7425	0.7121	5.3548	0.9513**	0.0010
KB Autosys Co Ltd	1.3184	301.3374	1.7094	1.3196	6.8080	0.9200**	0.0000
Austem Co Ltd	1.6799	189.7367	2.1202	0.3181	2.7821	0.9827	0.2142
AZTECH WB COMPANY LIMITED	1.9220	443.5969	1.6025	0.7829	6.3898	0.9342**	0.0001
Korea Real Estate Investment & Trust Co Ltd	0.9285	186.0678	1.9208	-0.1930	3.7473	0.9839	0.2639
Aurora World Corp	3.2377	262.7860	1.9382	0.5813	3.4991	0.9618**	0.0054
Atec Co Ltd	1.5973	297.7796	1.4837	2.3486	12.3691	0.8003**	0.0000
KODACO Co Ltd	0.1387	310.6263	1.8783	0.5002	4.0770	0.9785	0.1014
Komelon Corp	0.1290	83.1496	1.6696	0.4031	6.5324	0.9456**	0.0004
AnamInformationTechnology Corp	1.0806	480.9879	1.7874	0.2714	4.2859	0.9606**	0.0044
Kona I Co Ltd	2.0014	396.8853	1.9038	0.5366	4.3331	0.9740*	0.0448
AfreecaTV Co Ltd	2.5056	223.4037	1.8194	-0.0066	3.9551	0.9785	0.1009
KOSPI composite Index	0.7559	38.7713		-0.8696	5.6053	0.9566**	0.0023

Note: ** and * represent the hypothesis of normality is rejected respectively at 1% and 5% level of significance.

Table 4.4: Summary NYSE

Stock Name	Mean	Variance	Entropy	Skewness	Kurtosis	Shapiro-Wilk normality test	p-value
Hankuk Steel Wire Co Ltd	1.4061	385.6402	1.6333	0.5499	6.2177	0.9184**	0.0000
TaihanFiberoptics Co Ltd	0.1265	324.4033	1.6923	1.0514	6.5448	0.9298**	0.0000
Kocom Co Ltd	0.5341	223.9532	1.7166	0.9036	6.4239	0.9240**	0.0000
ATLASBX COMPANY LIMITED	3.5648	301.7039	1.7425	0.7121	5.3548	0.9513**	0.0010
KB Autosys Co Ltd	1.3184	301.3374	1.7094	1.3196	6.8080	0.9200**	0.0000
Austem Co Ltd	1.6799	189.7367	2.1202	0.3181	2.7821	0.9827	0.2142
AZTECH WB COMPANY LIMITED	1.9220	443.5969	1.6025	0.7829	6.3898	0.9342**	0.0001
Korea Real Estate Investment & Trust Co Ltd	0.9285	186.0678	1.9208	-0.1930	3.7473	0.9839	0.2639
Aurora World Corp	3.2377	262.7860	1.9382	0.5813	3.4991	0.9618**	0.0054
Atec Co Ltd	1.5973	297.7796	1.4837	2.3486	12.3691	0.8003**	0.0000
KODACO Co Ltd	0.1387	310.6263	1.8783	0.5002	4.0770	0.9785	0.1014
Komelon Corp	0.1290	83.1496	1.6696	0.4031	6.5324	0.9456**	0.0004
AnamInformationTechnology Corp	1.0806	480.9879	1.7874	0.2714	4.2859	0.9606**	0.0044
Kona I Co Ltd	2.0014	396.8853	1.9038	0.5366	4.3331	0.9740*	0.0448
AfreecaTV Co Ltd	2.5056	223.4037	1.8194	-0.0066	3.9551	0.9785	0.1009
KOSPI composite Index	0.7559	38.7713		-0.8696	5.6053	0.9566**	0.0023

Note: ** and * represent the hypothesis of normality is rejected respectively at 1% and 5% level of significance.

Table 4.5: Performance of different portfolio models (SSE)

			In-Sample					Out-of-Sample	
	ME	MEE	MV	MVE		ME	MEE	MV	MVE
Return	0.9565	0.9589	0.8703	0.8732		0.9306	0.9340	0.8264	0.8288
Variance	101.4074	101.8149	84.4549	84.4571		101.6361	101.7456	84.9182	84.9264
Entropy	0.1216	0.1219	0.2764	0.2731		0.1220	0.1221	0.2590	0.2576
SR(Var)	0.0949	0.0950	0.0947	0.0950		0.0923	0.0926	0.0897	0.0899
SR(Ent)	2.7440	2.7478	1.6591	1.6741		2.6643	2.6723	1.6238	1.6329
ASR	0.0938	0.0939	0.0934	0.0937		0.0913	0.0916	0.0886	0.0889
MADR	0.0965	0.0972	0.1153	0.1155		0.0925	0.0938	0.1058	0.1061
SSR	0.1323	0.1324	0.1295	0.1300		0.1286	0.1290	0.1234	0.1238
FTR	1.2713	1.2707	1.2881	1.2888		1.2632	1.2634	1.2670	1.2677
Portfolio turnover	0.0086	0.0032	0.0611	0.0615					

Note: SR(Var): Sharpe ratio based on variance as risk measure
SR(Ent): Sharpe ratio based on entropy as risk measure
ASR: adjusted Sharpe ratio
MADR: mean absolute deviation ratio
SSR: Sortino-Satchell ratio and
FTR: Farinelli-Ferreira-Rossello ratio

Table 4.6: Performance of different portfolio models (KRX)

			In-Sample					Out-of-Sample	
	ME	MEE	MV	MVE		ME	MEE	MV	MVE
Return	1.1352	1.1462	0.4525	0.4578		1.1015	1.1164	0.2447	0.2521
Variance	87.6614	87.6399	60.2256	60.2329		87.4116	87.2352	64.2148	64.0153
Entropy	0.1196	0.1201	0.7521	0.7317		0.1200	0.1201	0.9112	0.8877
SR(Var)	0.1205	0.1217	0.0580	0.0587		0.1178	0.1195	0.0305	0.0315
SR(Ent)	3.2817	3.3074	0.5669	0.5803		3.1803	3.2211	0.2564	0.2676
ASR	0.1193	0.1204	0.0578	0.0584		0.1159	0.1175	0.0307	0.0316
MADR	0.1358	0.1349	0.0691	0.0699		0.1314	0.1322	0.0386	0.0391
SSR	0.1814	0.1827	0.0829	0.0840		0.1752	0.1775	0.0438	0.0452
FTR	1.3759	1.3787	1.1682	1.1702		1.3639	1.3685	1.0885	1.0912
Portfolio turnover	0.0095	0.0035	0.1131	0.1131					

Note: SR(Var): Sharpe ratio based on variance as risk measure
SR(Ent): Sharpe ratio based on entropy as risk measure
ASR: adjusted Sharpe ratio
MADR: mean absolute deviation ratio
SSR: Sortino-Satchell ratio and
FTR: Farinelli-Ferreira-Rossello ratio

Table 4.7: Performance of different portfolio models (NYSE)

			In-Sample					Out-of-Sample	
	ME	MEE	MV	MVE		ME	MEE	MV	MVE
Return	0.5417	0.5468	0.5016	0.4982		0.5384	0.5415	0.4333	0.4333
Variance	32.5262	32.0207	21.2564	21.2683		32.6766	32.2088	22.0416	22.0416
Entropy	0.1183	0.1189	0.3762	0.3606		0.1187	0.1192	0.3277	0.3277
SR(Var)	0.0950	0.0967	0.1088	0.1080		0.0942	0.0954	0.0923	0.0923
SR(Ent)	1.5751	1.5861	0.8169	0.8286		1.5629	1.5686	0.7570	0.7570
ASR	0.0925	0.0949	0.1078	0.1070		0.0925	0.0937	0.0917	0.0917
MADR	0.1423	0.1383	0.1181	0.1206		0.1423	0.1357	0.0999	0.0999
SSR	0.1241	0.1278	0.1568	0.1554		0.1241	0.1262	0.1322	0.1322
FTR	1.2915	1.2980	1.3216	1.3200		1.2915	1.2934	1.2678	1.2678
Portfolio turnover	0.0061	0.0023	0.1078	0.1017					

Note: SRVar): Sharpe ratio based on variance as risk measure
SR(Ent): Sharpe ratio based on entropy as risk measure
ASR: adjusted Sharpe ratio
MADR: mean absolute deviation ratio
SSR: Sortino-Satchell ratio and
FTR: Farinelli-Ferreira-Rossello ratio

Summary statistics of stock returns of Shanghai stock exchange (SSE), Korea exchange (KRX), New York stock exchange (NYSE) are displayed in Table 4-4. Most of the stock returns are non-normal, asymmetric and have excess kurtosis. Ranges of stock return variances are different from that of conditional entropy subject to the market index. This difference is not unexpected since the data are non-normal and that entropy also depends on higher order moments compared to variance.

Performance measures of portfolio models for SSE are reported in Table 4.5. The highest portfolio returns are given by MEE in case of both in- and out-of-sample periods. The lowest in-sample variance is given by MV accompanied by MVE, whereas, the lowest out-of-sample variance is obtained from solely MV. The in- and out-of-sample entropies are the lowest for MEE and ME. The highest in-sample SR (Variance) is given by MEE and ME as well, whereas, out-of-sample SR (variance) is highest for solely MEE. The highest in-and out-of-sample SR (Entropy) is obtained from MEE. The highest in-sample ASR is given by MEE, whereas the highest out-of-sample ASR is obtained from both ME and MEE. Both in- and out-of-sample MADR is the highest for MV and MVE. Both in- and out-of-sample SSR is given by MEE and the highest in-sample FTR is from MVE and the highest out-of-sample FTR is obtained from MVE. The lowest portfolio turnover is obtained from MEE. As whole, multi-objective models, MEE and MVE, perform better than single-objective models, ME and MV. More specifically, MEE offers the highest in- and out-of-sample portfolio returns and performs better than the other models in most of the cases.

Performance measures of portfolio models for KRX are reported in Table 4.6. The highest portfolio return for both in- and out-of-sample periods is given by MEE. The lowest in-sample variance is given by MV accompanied by MVE. The lowest out-of-sample variance is given by MVE. The lowest in- and out-of-sample entropy is given by

MVE. Both in- and out-of-sample SR (Variance) is the highest for MEE and ME. Both the highest in- and out-of-sample SR (Entropy) is obtained from MEE. The highest in- and out-of-sample ASR is given by MEE and the highest in-sample MADR is given by ME and out-of-sample MADR is given by MEE. The highest In-sample SSR is obtained from MEE and ME, whereas, the highest out-of-sample SSR is given by MEE. The highest in-and out-of-sample FTR is in favor of MEE and ME. The lowest portfolio turnover is obtained from MEE. As a whole, MEE offers the highest in- and out-of-sample portfolio return and most of the performance measures show it has better in- and out-of-sample performance.

Performance measures of portfolio models for NYSE are reported in Table 4.7. Both the highest portfolio returns in- and out-of-sample periods is given by MEE. The lowest in-sample variance is given by MV. The lowest out-of-sample variance is given by MVE accompanied by MV. The lowest in- and out-of-sample entropy is given by ME. In-sample SR (Variance) is the highest for MV and the highest out-of-sample SR (Variance) is MEE. Both the highest in- and out-of-sample SR (Entropy) is obtained from MEE. The highest in-sample ASR is given by MV and the highest out-of-sample ASR is given by MEE. Both the highest in- and out-of-sample MADR is given by ME. Both the highest in-and out-of-sample SSR is obtained from MV and the highest in-sample FTR is in favor of MV and the highest out-of-sample FTR is given by MEE. The lowest portfolio turnover is obtained from MEE. As a whole, MEE offers the highest in- and out-of-sample portfolio return and most of the performance measures show it has better in- and out-of-sample performance.

4.5. Summary

We observe that stock returns are non-normal, asymmetric and have excess kurtosis and ruled by uncertainty; hence, diversity of the portfolio is demanded. In the early

literature, entropy is used independently either as an alternative risk measure or as a means of achieving diversity. In MEE, the multi-purpose use of entropy in the same model offers a more complete portfolio model: entropy of historical returns provides a nonparametric, thus less restricted, risk measure, and at the same time, inclusion of entropy of weights in the objective function ensures a desired level of diversity. Our simulation results support that MEE is most diversified and maintain a better performance compared to other models. Entropy based multi-objective model is further evaluated and compared with single-objective portfolio in context of a wide range of empirical data set. A rolling window procedure is used to assess the in- and out-of-sample performances of asset allocation models. Evaluation on different equity market data reveals that a multi-objective model with entropy risk offers higher in- and out-of-sample portfolio return as well as higher out-of-sample Sharp ratio than other models considered in this study. MEE also has lower transaction cost (portfolio turnover). The out-of-sample results of stock market data indicate that the potentially large investment gain can be realized using MEE in place of several approaches for portfolio invest.

CHAPTER 5: ROBUST ENTROPY ESTIMATION FOR PORTFOLIO ANALYSIS

5.1 Introduction

The presence of outlier and non-normality are the major problem of decision making based on historical data. Portfolio performance mostly depends on asset allocation strategy. The Markowitz portfolio optimization estimates the expected return and the covariance matrix from historical return time series and treats them as true parameters for portfolio selection. The simple sample mean and covariance matrix are used as the parameters since they are the best unbiased estimators under the assumption of multivariate normality. However, mean and covariance are sensitive to outlier, and even small changes in these estimates can lead to a significant change in the composition of the efficient frontier. This naive certainty equivalence mean-variance approach, thus, often leads to extreme portfolio weights instead of a diversified portfolio as the method anticipates and dramatic swings in weights when there is a minor change to the expected returns or the covariance matrix (Dickenson, 1974; Jorion, 1986 and Klein et al., 1979). This may lead us to frequently and mistakenly rebalance our portfolio to stay on this elusive efficient frontier, incurring unnecessary transaction costs. The problem is further exacerbated if the number of observations is of the same order as the number of assets, which is often the case in financial applications to select industry sectors or individual securities. Nevertheless, the original form of mean-variance portfolio optimization has rarely been applied in practice because of this drawback.

A number of alternative models have been developed to improve parameter estimation. For example, factor-based models try to reduce the model complexity (number of parameters) by explaining asset return variances/covariances using a limited number of common factors. Multivariate GARCH models try to address fat tails and volatility

clustering by incorporating the time dependence of returns in the covariance matrix but neither approach effectively reduces or eliminates the influences of outliers in the data. A small percentage of outliers, in some cases even a single outlier, can distort the final estimated variance and covariance.

Evidence has shown that the most extreme (large positive or negative) coefficients in the estimated covariance matrix often contain the largest error and as a result, mean-variance optimization based on such a matrix routinely gives the heaviest weights either positive or negative to those coefficients that are most unreliable. This "error-maximization" phenomenon (Michaud, 1989) causes the mean-variance technique to behave very badly unless such errors are corrected.

In this chapter, we focus on investigating robust statistical approaches to reduce the influence of outliers, to increase the stability of the portfolio. We first examine the sensitivity of risk measures such as variance or entropy due to outliers and found that both variance and entropy are sensitive to outlier. We, therefore, propose a new robust estimator for kernel density that is eventually used for robust estimation of entropy and portfolio analysis as well. Our simulation results show that use of proposed robust measure of risk in portfolio analysis can render the effect of contamination.

5.2 Basic concept of Robustness

Outlier

An outlier is an observation that lies an abnormal distance from other values in a random sample from a population. Barnett and Lewis (1995) define an outlier to be an observation or subset of observations which appears to be inconsistent with the remainder of the dataset. Outliers are sometimes referred to as contaminants. Outliers have different sources: they may be the result of a recording and measuring errors, they

may arise from the inherent variability of the dataset (i.e. extreme values from the tails of the distribution) or they may be generated from another model. Outliers may be univariate or multivariate. Multivariate outliers are observations that are inconsistent with the correlational structure of the dataset. Thus, while univariate outlier detection is performed independently on each variable, multivariate methods investigate the relationship of several variables.

Masking and Swamping

Barnett and Lewis (1994) define masking as the tendency for the presence of extreme observations not declared as outliers to mask the discordancy of more extreme observations under investigation as outliers. Swamping refers treating clean observations as outliers mistakenly. Masking can occur when we specify too few outliers in the test, for example, if we are testing for a single outlier when they are in fact two (or more) outliers, the additional outlier may influence the value of the test statistic enough so that no points are declared as outlier. On the other hand, swamping can occur when we specify too many outliers in the test. For example, if we are testing for two or more outliers when they are in fact only a single outlier, both points maybe declared outlier.

Measure of Robustness

There are several measures to determine the robustness of an estimator. The breakdown point of an estimator is the largest fraction of the data that can be moved arbitrarily without perturbing the estimator to the boundary of the parameter space thus the higher the breakdown point, the more robust the estimator against extreme outliers. However, the breakdown point is not enough to assess the degree of robustness of an estimator.

A natural way to assess the stability of an estimator is to make a sensitivity analysis and a simple way to do this is to obtain the influence function (IF) or compute the sensitivity curve (SC). The influence function (Costa and Deshayes 1977, Deniau et al. 1977, Huber 1981, Hampel et al. 1986) of an estimator $T = T(x_1, \dots, x_n)$is defined as

$$IF(y;T,F) = \lim_{\varepsilon \downarrow 0} \frac{T((1-\varepsilon)F + \varepsilon\delta_y) - T(F)}{\varepsilon} \tag{5.1}$$

for any $y \in \mathbb{R}$ such that the limit exists. $IF(y;T,F)$ measures the sensitivity of the risk estimator T to the addition of a new data point in a large sample. An alternative to IF is the sensitivity curve (SC), which can be defined as

$$SC_n(y) = n(T_n(x_1, \dots, x_{n-1}, y) - T_{n-1}(x_1, \dots, x_{n-1})), \tag{5.2}$$

where $T_n(x)$denotes the estimator of interest based on the sample x of size n. The sensitivity curve $SC_n(y)$ is a translated and rescaled version of the empirical influence function. In many situations, $SC_n(y)$ will converge to the influence function when $n \to \infty$.

The Gross Error Sensitivity (g.e.s.) expresses asymptotically the maximum effect a contaminated observation can have on the estimator. It is the maximum absolute value of the IF. The asymptotic bias of an estimator is defined as the maximum effect of the contamination of a given distribution with a proportion from an outlying distribution. Instead of breakdown point, the gross error sensitivity gives an exact measure of the size of robustness, since it is the supremum of the influence function of an estimator, and it is a measure of the maximum effect an observation can have on an estimator. Details of robust statistics and their measures of assessment are available in Huber (1981); Hampel et al. (1986); Rousseeuw and Leroy (1987); Staudte and Sheather (1990).

5.3 Sensitivity of outlier on risk estimation

Methods of robust statistics are known to be relevant in quantitative finance. Statistical estimation and sensitivity analysis of risk measures have been studied by Gourieroux et al.(2000) and Gourieroux and Liu (2006). In particular, Gourieroux and Liu (2006) consider non-parametric estimators of distortion risk measures and focus on the asymptotic distribution of these estimators.

The presence of outliers leads to distorted estimates of the population mean and the dispersion. It is, therefore, important that these unusual observations in both tails of the distribution be treated adequately. Two simple robust estimators of location and scale parameters are the median and the MAD (the median absolute deviation), respectively. However, median and MAD are not efficient estimator since they are not based on all observations. For full uncensored data sets, simple robust estimates such as the trimmed mean or Winsorized mean (Hoaglin, Mosteller, and Tukey, 1983) are sometimes used to estimate the population mean in the presence of outliers. For example, a $100p\%$ trimmed mean is obtained by using only the middle $n(1-2p)$data values and the np values are omitted from each of the two (left and right) tails of the data set.

A good compromise between robustness and efficiency can be obtained with M-estimates. If σ is known, an M-estimate of μ is implicitly defined as a solution of the estimating equation

$$\sum_{\mathbf{i=1}}^{\mathbf{n}} \boldsymbol{\Psi}_{\mathbf{k}}\left(\frac{\mathbf{x_i}-\boldsymbol{\mu}}{\boldsymbol{\sigma}}\right) = \mathbf{0}$$

where $\Psi_k(.)$ is a suitable function. Huber (1981) suggests

$$\Psi_k(x) = max(-k, min(k, x)) \qquad (5.3)$$

An M-estimate of mean can be written as $\sum_{i=1}^{n} w_i x_i$, where $w_i = W\left(\frac{x_i}{\sigma}\right)$ is a weight $i = 1, \ldots, n$ with $W(u)$ suitable weight function. For example, for the Huber-type estimate

$$W(u) = \Psi_{\mathrm{k}}(u)/u \tag{5.4}$$

with $\Psi_{\mathrm{k}}(.)$ given in (5.3).

Figure 5.1 shows the sensitivity of variance and entropy for a single outlier of different sizes. As we observe, the traditional estimates of both variance and entropy are affected by outliers largely. On the contrary, the robust methods (trimming, Huber) provide estimators that are less affected by outliers. These preliminary results suggest using robust procedure for portfolio risk estimation.

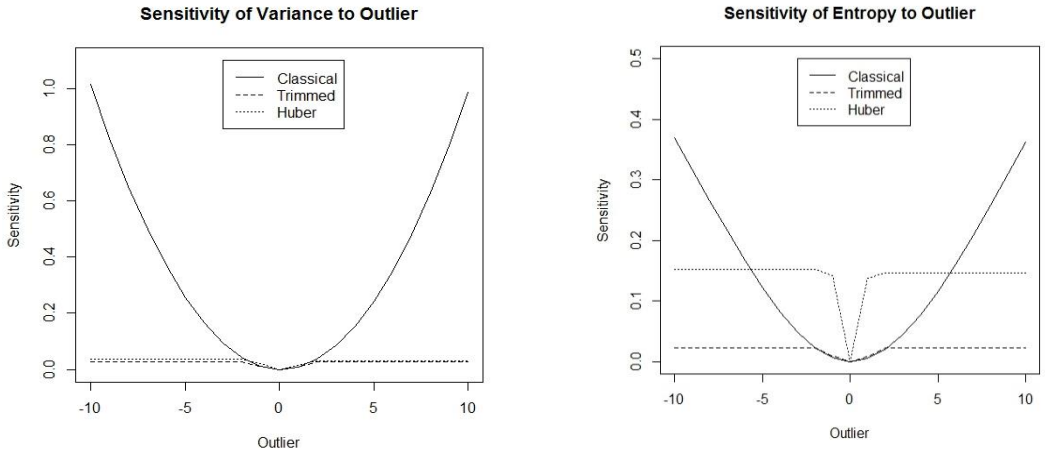


Figure 5.1: Sensitivity Curve of Variance and Entropy

5.4 Multivariate outlier and Mahalanobis' Distance

Mahalanobis' distance can be thought of as a metric for estimating how far each case is from the center of all the variables' distributions (i.e. the centroid in multivariate space).

For a d-dimensional multivariate sample x_i $(i = 1,\ldots,n)$ the Mahalanobis' distance is defined as

$$MD_i = \sqrt{(x_i - \mu)^T V^{-1}(x_i - \mu)}, \tag{5.5}$$

where μ is the multivariate location and V the covariance matrix.

A classical way for multivariate outlier detection is to compute Mahalanobis' distance (Mahalanobis, 1927; 1936). Using estimates of the location and variation, this distance identify observations that are isolated from the main stream of data. Multivariate outliers can now simply be defined as observations having a large (squared) Mahalanobis' distance. For this purpose, a quantile of the χ^2distribution (e.g., the 97.5% quantile) could be considered. With the assumption of multivariate normal distribution (*d* dimensions), Mahalanobis distance of sample data follows a χ^2 distribution with *d* degrees of freedom. The standard method for multivariate outlier detection is−robust estimation of the parameters in the Mahalanobis' distance and the comparison with a critical value of the χ^2distribution (Rousseeuw and Van Zomeren, 1990). However, also values larger than this critical value are not necessarily outliers; they could still belong to the data distribution. A better procedure than using a fixed threshold is to adjust the threshold to the data set at hand. In order to distinguish between extremes of a distribution and outliers, Garrett (1989) introduced the χ^2plot, which draws the empirical distribution function of the robust Mahalanobis distances against the χ^2distribution. A break in the tail of the distributions is an indication for outliers, and values beyond this break are iteratively deleted.

The Mahalanobis distances need to be estimated by a robust procedure in order to provide reliable measures for the recognition of outliers. Single extreme observation or groups of observations, departing from the main data structure can have a severe

influence to this distance measure, because both location and covariance are usually estimated in a non-robust manner. Many robust estimators for location and covariance have been introduced in the literature. The minimum covariance determinant (MCD) estimator is probably most frequently used in practice, partly because a computationally fast algorithm is available (Rousseeuw and Van Driessen, 1999). Using robust estimators of location and scatter in (5.5) leads to so-called robust distances (RD). Rousseeuw and Van Zomeren (1990) used these RDs for multivariate outlier detection. If the squared RD for an observation is larger than, say, $\chi^2_{d,0.975}$,it can be declared a candidate outlier.

When using R there are multiple ways of calculating the Mahalanobis distance of a given data set. One way is using the *chemometric* package (Filzmoser and Varmuza, 2013). The chemometric package contains a function *Moutlier* for calculating and plotting both the Mahalanobis' distance and a robust version of the Mahalanobis' distance. The robust Mahalanobis' distance is based on the minimum covariance determinant (MCD) estimate. The robust Mahalanobis distance can also be obtained using robust package with covRob function. In order to obtain a good estimate in a reasonable amount of time, this function choose either Stahel-Donoho, Fast MCD or Orthogonalized Gnanadesikan-Kettenring for computing robust covariance depending on dimension and size of data.

Fast MCD

The general principle of robust statistical estimation is to give full weights to observations assumed to come from the main body of the data, but to reduce or completely eliminate weights for the observations from tails of the contaminated data. The minimum covariance determinant (MCD) method, a robust estimator introduced by Rousseeuw in 1985, eliminates perceived outliers from the estimation of the mean and

the covariance matrix. It uses the mean and the covariance matrix of hdata points $\left(\frac{\mathrm{T}}{2} \leq \mathrm{h} < \mathrm{T}\right)$with the smallest determinantto estimate the population mean and the covariance matrix. The method has a breakdown value of $\frac{(\mathrm{T}-\mathrm{h})}{\mathrm{T}}$. If the data come from a multivariate normaldistribution, the average of the optimal subset is an unbiased estimator of the population mean. The resulting covariance matrix is biased, but a finite sample correction factor $(\mathrm{c}_{\mathrm{h,T}} \geq 1)$ can be used to make the covariance matrix unbiased. The multiplication factor $\mathrm{c}_{\mathrm{h,T}}$ can be determined through Monte-Carlo simulation. For our specific purpose, the bias by itself does not affect the asset allocations in all pairs of covariances are underestimated by the same factor.

MCD has rarely been applied to high-dimensional problems because it is extremely difficult to compute. MCD estimators are solutions to highly nonconvex optimization problems that have exponential complexity of the order 2^N in terms of the dimension N of the data. Therefore, these original methods are not suitable for asset allocation problems when $\mathrm{N} > 20$. Yet, in practice, assetallocation problems often include dozens of industrial classes or hundreds of individual securities, which makes the MCD method computationally infeasible.

In order to cope with computational complexity problems, a heuristic FAST-MCD algorithm developed by Rousseeuw and Van Driessen (1999), provides an efficient alternative. A naive MCD approach would compute the MCD for up to $\binom{T}{h}$ subsets, while FAST-MCD uses sampling to reduce the computation and usually offers a satisfactory heuristic estimation. The key step of the FAST-MCD algorithm takes advantage of the fact that, starting from any approximation to the MCD, it is possible to compute another approximation with a determinant no higher than the current one. The methods based on the following theorem related to a concentration step (C-step):

Let $H_1 \subset \{1,2,\dots,n\}$ be any h-subset of the original cross-sectional data, put $\hat{\mu}_1 = \frac{1}{h}\sum_{t\in H_1} R_t$ and $\hat{\Sigma}_1 = \frac{1}{h}\sum_{t\in H_1}(R_t - \hat{\mu}_1)(R_t - \hat{\mu}_1)'$. If $\det(\hat{\Sigma}_1) \neq 0$, define the distance $d_1(t) = \sqrt{(R_t - \hat{\mu}_1)(R_t - \hat{\mu}_1)'}$, t=1,2,…,T. Take H_2 such that $\{d_1(i);\ i \in H_2\} = \{(d_1)_{1:T}, \dots, (d_1)_{h:T}\}$ where $(d_1)_{1:T} \leq (d_1)_{2:T} \leq \cdots \leq (d_1)_{T:T}$ are the ordered distributions, and compute $\hat{\mu}_2$ and $\hat{\Sigma}_2$ based on H_2. Then $\det(\hat{\Sigma}_2) \leq \det\hat{\Sigma}_1$ with equality if and only if $\hat{\mu}_2 = \hat{\mu}_1$ and $\hat{\Sigma}_1 = \hat{\Sigma}_2$.

If $\det(\hat{\Sigma}_1) > 0$, the C-step yields $\hat{\Sigma}_2$ with $\det(\hat{\Sigma}_2) \leq \det\hat{\Sigma}_1$. Basically, the theorem indicates the sequence of determinants obtained through C-steps converges in a finite number of steps from any original h-subset to a subset satisfying $\det\left(\hat{\sum}_{m+1}\right) = \det(\sum_m)$. Afterward, running the C-step no longer reduces the determinant. However, this process only guarantees that the resulting $\det(\hat{\sum})$ is a local minimum instead of the global one. To yield the h-subset with global minimum $\det(\hat{\sum})$ or at least close to optimal, many initial choices (often >500) of H_1 are taken and C-steps are applied to each.

Stahel-Donoho projection based estimator

The first affine equivariant multivariate location estimator robust enough to tolerate up to 50% of outliers in the sample before it breaks down was independently discovered by Stahel (1981) and Donoho (1982). They proposed to solve the dimensionality problem by computing the weights for the robust estimators from the projections of the data onto some directions. These directions were chosen to maximize distances based on robust univariate location and scale estimators, and the optimal values for the distances could also be used to weight each point in the computation of a robust covariance matrix.

Let X be an $n \times p$ data matrix that contains n observations $x_1, x_2, \dots, x_n$. Let μ and σ be shift and scale equivarentunivariate location and scale statistics. Then for any $y \in \mathbb{R}^p$, the Stahel-Donohooutlyingness is defined as

$$T_{SD} = \frac{\sum_{i=1}^{n} w_i x_i}{\sum_{i=1}^{n} w_i},$$

and

$$S_{SD} = \frac{\sum_{n}^{i=1} w_i (x_i - T_{SD})(x_i - T_{SD})'}{\sum_{i=1}^{n} w_i},$$

where $w_i = w(r_i)$ and $w: \mathbb{R}^+ \to \mathbb{R}^+$ is a weight function so that observation with large outlyingness get small weights (see Stahel,1981; Donho,1982).

To ensure a high breakdown point, one global optimization problem with discontinuous derivatives had to be solved for each data point, and the associated computational cost became prohibitive for large high-dimensional datasets. This computational cost can be reduced if the directions are generated by a resampling procedure of the original data, but the number of directions to consider still grows exponentially with the dimension of the problem.

Orthogonalized Gnanadesikan-Kettenring pairwise estimator

The orthogonalized Gnanadesikan-Kettenring (OGK) estimator, proposed by Marrona and Zamar (2002), is a modified version of Gnanadesikan-Kettenring robust covariance estimate. The authors argue that performance of OGK is comparable to that of the Stahel-Donoho (SD) and better than fast MCD (FMCD) estimates. Moreover, it is much faster than both, especially for large dimension.

Let $x_1, x_2 \ldots, x_n \in \mathbb{R}^p$ be a dataset. Let $\sigma(.)$ and $\mu(.)$ be robust univariate dispersion and locationstatistics, and let $V(.,.)$ be a robust estimate of the covarianceof two random variables. Define a scatter matrix $V(X)$anda location vector t(X) as follows:

1. Let D=diag $\left(\sigma(X_1), \ldots, \sigma(X_p)\right)$ and $y_i = D^{-1}x_i,\ i = 1,2, \ldots, n.$

2. Compute the "correlation matrix"$U = [U_{jk}]$, applyingGnanadesikan–Kettenring estimator ν to the columns of Y, that is

$$U_{jj} = 1 \text{ and } U_{jk} = \frac{1}{4}[\sigma(Y_j + Y_k)^2 - \sigma(Y_j - Y_k)^2], \qquad j \neq k.$$

3. Compute the eigenvalues λ_j and eigenvectors $e_j$of U $(j \equiv 1,2, \ldots, p)$ and call E the matrix whose columns are the $e_{j'}$s, so that $U \equiv E \wedge E'$,where $\wedge = diag(\lambda_1, \lambda_2, \ldots, \lambda_p)$

4. Define $A = DE$ and $z_i = E'y_j = A^{-1}x_i$ so that $x_i = Az_i$

Then the location and dispersion estimate is computed as

$$t(X) = A\nu \text{ and } V(X) = A\Gamma A'$$

where $\Gamma = diag(\sigma(Z_1)^2, \ldots, \sigma(Z_p)^2)$ and $\nu = (\mu(Z_1), \ldots, \mu(Z_p))'$

The OGK estimator will be obtained by an iterative reweighted process.

5.5 Robustness of Entropy Estimation

Recall the Shanon entropy for variable X with *pdf*

$$H_X = -\int p(x) \ln p(x)\, dx$$

For a plug in estimator we first need an empirical estimation of $p(x)$, say $\hat{p}(x)$. A histogram or a kernel density estimator usually discretize the distribution and provide

estimates of probability at some discrete points over the range of the variable X. Let X be discretized at $(x_1, x_2, \dots, x_m)$ with corresponding probability $(p(x_1),\ p(x_2), \dots, p(x_m))$. The estimator of H_X is thus

$$\widehat{H}_{x_m} = -\sum_{i=1}^{m} p(x_i) \ln p(x_i) \tag{5.6}$$

The value of $\widehat{H}_x$ will be heavily dependent on discretization. If $m > l$, it is obvious that $\widehat{H}_{x_m} \geq \widehat{H}_{x_l}$. Therefore, the selection of number of bins for a histogram or the number of evaluation points of a kernel density is crucial. Too many evaluation points may results over estimate while too small number of evaluation points may results under estimate of the density.

The second important issue in entropy calculation is that too small value of $p(x)$ may largely affect the estimate of H_X since H_X is directly estimated from $p(x)$. In turn, entropy could be wrongly estimated if $p(x)$ takes value close to zero. Small value of $p(x)$ usually comes from the tail area. Specifically, if the data contain outlier, a good density estimator (specially, a robust density estimator) should provide small value of $p(x)$ for the corresponding outlier. Therefore, presence of outliers can affect the estimate of entropy. A precaution can be taken in computing entropy from real data:

$$\underset{p(x)\to 0}{H_X} \cong 0$$

This will results in truncated entropy such as for given a small positive quantity $\alpha > 0$,

$$\widehat{H}_X(trancate) = \sum_{p(x)\geq\alpha} p(x_i) \log p(x_i) \tag{5.7}$$

We, thus, suggest using a robust density estimator that is not affected by outlier and at the same time adequately select the number of bins (for histogram) or the number of

grid points (for kernel density). Additionally, during computation of entropy, small values of the probability estimate should be avoided.

5.6 Adaptive Robust Kernel Density Estimator

The classical kernel estimator is introduced by (Parzen, 1962 and Rosenblatt, 1956). Epanechnikov (1969) shows that there exists an optimal kernel in the sense of the asymptotic integrated square error (AMEX), which is a part of a parabola. In fact any reasonable kernel gives results that are close to optimal. Kernel density estimation is a very useful tool for exploring the distribution structure of unknown population (Park and Marron, 1990). For computational convenience, the standard normal kernel or its Fast Fourier transform (Silverman, 1986) is often used. There are several studies on bandwidth selection for kernel density. Surveys on the most interesting approaches for kernel density estimation and their variants are available in Bowman (1985) or Silverman (1986). The computational issue of kernel density has been addressed by Wand and Jones (1994) and Duong and Hazelton, (2003; 2005). Furthermore, there are comparative studies of some of these methods in Scott and Factor (1981), Bowman (1985) and Kappenman (1987). M-estimation applied to kernel density has been studied by Kim and Scott, (2012), and Demitri and Zoubir (2014). The proposed method achieves robustness by combining a traditional kernel density estimator with M-estimation for the mean of the kernel. They argue that the kernel density is sensitive to outlier and can be improved if robust procedure is adapted.

We propose a robust version of the kernel density estimator, the weights for the kernel density is selected by a robust estimate of Mahalanobis distance linked with β-divergent principal (Higuchi and Eguchi, 2004; Mollah et al., 2010). The weight function that we propose has the form

$$\psi_\beta(x,\mu,V) = exp(-\beta w(x,\mu,V)) \tag{5.8}$$

with

$$w(x,\mu,V) = \frac{1}{2}(x-\mu)^T V^{-1}(x-\mu) \tag{5.9}$$

$$= \frac{1}{2}MD^2 \text{ ,} \quad \text{a half of the squared Mahalanobis distance,}$$

where μ and V are the mean vector and covariance matrix of x. The procedure of weight calculation and selection of the tuning parameter β will be discussed in the next section.

The β-divergent principal is a highly robust procedure, and has been applied in principal component analysis (Highuchi and Eguchi, 2004; Mollah et al., 2010) and hierarchical clustering (Badsha et al., 2013). Mollah et al. (2012) examine the influence function and sensitivity curve of β-divergent estimator. They found that with proper choice of β, both IF and SC of this estimator are bounded and, thus, β-divergent estimator is B-robust.

Consider the density functions $p(x)$ and $q(x)$ defined on a d-dimensional data space, $\mathbb{R}^d$. The β-divergence of $p(x)$ with respect to $q(x)$ is defined as

$$D_\beta(p,q) = \int \left[\frac{1}{\beta}\{p^\beta(x) - q^\beta(x)\}p(x) - \frac{1}{\beta+1}\{p^{\beta+1}(x) - q^{\beta+1}(x)\}\right]dx \text{ for } \beta > 0 \tag{5.10}$$

which is non-negative, that is $D_\beta(p,q) \geq 0$, equality holds if and only if $p(x) = q(x)$ for almost all x in $\mathbb{R}^d$, see Basu et al. (1998) and Minami and Eguchi (2002), for example. Note that when the tuning parameter β tends to 0, β-divergence becomes Kullback-Leiber (KL) divergence such as

$$\lim_{\beta \downarrow 0} D_\beta(p,q) = \int p(x) \log \frac{p(x)}{q(x)} dx = D_{KL}(p,q)$$

If $p(x)$ be the density function for the variable of interest x, then the minimum β-divergence procedure is defined by

$$\min_{q\in M} D_\beta(p,q),$$

where M denotes a statistical model. Consider a kind of volume match by

$$D_\beta^*(p,q) = \min_k D_\beta(p,kq)$$

$$= \frac{1}{\beta(\beta+1)}\left[p^{\beta+1}(x)dx - \frac{\{\int p(x)q^\beta(x)dx\}^{\beta+1}}{\{\int q^{\beta+1}(x)dx\}^\beta}\right]$$

For a fixed data density p the functional $D_\beta^*(p,.)$ is defined on the space of nonnegative functions with a finite mass and $D_\beta^*(p,kq) = D_\beta^*(p,q)$ for any positive scalar k. If the first terms that depend only on p isneglected, it will be of the form

$$-\frac{\{\int p(x)q^\beta(x)dx\}^{\beta+1}}{\{\int q^{\beta+1}(x)dx\}^\beta}$$

which is monotonically transformed into

$$-\frac{\int p(x)q^\beta(x)dx}{\{\int q^{\beta+1}(x)dx\}^{\frac{\beta}{\beta+1}}}$$

This will provide a linear functional on p as

$$L_\beta(q;p) = \frac{1}{\beta}\left[1 - \frac{\int p(x)q^\beta(x)dx}{\{\int q^{\beta+1}(x)dx\}^{\frac{\beta}{\beta+1}}}\right]$$

By definition

$$\underset{q\in M}{\operatorname{argmin}}\, D_\beta{}^*(p,q) = \underset{q\in M}{\operatorname{argmin}}\, L_\beta{}^*(p,q)$$

for any statistical model *M* of density function. We observe that

$$\lim_{\beta\downarrow 0} L_\beta(q;p) = -\int p(x)\log q(x)dx$$

which is the expected log-loss function, or minus the expected log likelihood function. Let us consider this β-divergence $D_\beta{}^*(p,q)$ or $L_\beta(p,q)$ in which the model function is a Gaussian density function $\varphi_{\mu,V}(x)$ with the mean vector μ and variance matrix V. Then the minimum β-divergence estimators for μ and V are obtained by minimization of $D_\beta{}^*\left(p,\varphi_{\mu,V}\right)$ or equivalently, the minimum β-divergence estimators are derived by minimization of

$$L_\beta(\mu,V;p) = \frac{1}{\beta}\left[1 - det(V)^{-\frac{1}{2}\frac{\beta}{\beta+1}} \times \int exp\{-\beta w(x,\mu,V)\}\,p(x)dx\right] \tag{5.11}$$

since $L_\beta(\mu,V;p) \equiv L_\beta\left(\varphi_\mu,v;p\right)$where *w* is defined in (5.9).

The expected β-loss function has the empirical form $L_\beta(\mu,V)$:

$$L_\beta(\mu,V) = \frac{1}{n}\sum_{t=1}^{n}\frac{1}{\beta}[1 - \det(V)^{\frac{-1}{2}\frac{\beta}{\beta+1}} \exp\{-\beta w(\mathrm{x_t},\mu,\mathrm{V})\}] \tag{5.12}$$

Similarly, we find another form equivalent to $L_\beta(\mu,V;p)$ as

$$L_\beta{}^*(\mu,V;p) = (\beta+1)\log\left\{\int p(x)\varphi_{\mu,V}^{\beta}dx\right\} - \beta\log\left\{\varphi_{\mu,V}^{\beta+1}(x)dx\right\} \tag{5.13}$$

If a gradient of $L_\beta(\mu,V)$ with respect to (μ,V) is taken, the minimizer of $L_\beta(\mu,V)$ will be obtained. It is equivalent to solving the equations $\mu^* = \mu$ and $V^* = V$ in the following:

$$\mu^* = \frac{\sum_{t=1}^{n}\psi_\beta(x_t,\mu,V)x_t}{\sum_{t=1}^{n}\psi_\beta(x_t,\mu,V)} \tag{5.14}$$

and

$$V^* = (\beta + 1)\frac{\sum_{t=1}^{n} \psi_\beta(x_t,\mu,V)(x_t-\mu)(x_t-\mu)^T}{\sum_{t=1}^{n} \psi_\beta(x_t,\mu,V)} \tag{5.15}$$

where

$$\psi_\beta(x,\mu,V) = exp\{-\beta w(x,\mu,V)\} \tag{5.16}$$

$$w(x,\mu,V) = \frac{1}{2}(x-\mu)^T V^{-1}(x-\mu) \tag{5.17}$$

Note that $w(x,\mu,V)$ is nothing but half of the squared Mahalanobis distance (MD), which is a popular and well accepted measure for identifying outlier. Eventually, the weight $w(x,\mu,V) = \frac{1}{2}MD^2$, thus, becomes a function of univariate variable, MD from a function of multivariable variable x. In our analysis we use robust version of MD for calculating the weights.

The estimation of μ and V can be obtained iteratively as

$$\mu_{j+1} = \frac{\sum_{t=1}^{n} \psi_\beta(x_t|\mu_j,V_j)x_t}{\sum_{t=1}^{n} \psi_\beta(x_t|\mu_j,V_j)} = \frac{\sum_{t=1}^{n} \psi_\beta(x_t|MD_j)x_t}{\sum_{t=1}^{n} \psi_\beta(x_t|MD_j)} \tag{5.18}$$

$$V_{j+1} = (\beta + 1)\frac{\sum_{t=1}^{n} \psi_\beta(x_t|\mu_j,V_j)(x_t-\mu_j)(x_t-\mu_j)^T}{\sum_{t=1}^{n} \psi_\beta(x_t|\mu_j,V_j)} \tag{5.19}$$

$$= (\beta + 1)\frac{\sum_{t=1}^{n} \psi_\beta(x_t|MD_j)(x_t-\mu_j)(x_t-\mu_j)^T}{\sum_{t=1}^{n} \psi_\beta(x_t|MD_j)}$$

Selection of β

We observe that the performance of the proposed method for robust kernel depends on the value of the tuning parameterβ. To ensure better performance by this method, β should be selected with an adaptive selection procedure depending on data structure. To find an appropriate β, a number of trial values should be evaluated.

Define a measure for evaluating the estimators for μ and V as

$$D_{\beta_0}(\beta) = E\{L_{\beta_0}(\hat{\mu}_\beta, \hat{V}_\beta)\}$$

where

$$(\hat{\mu}_\beta, \hat{V}_\beta) = \underset{\mu,V}{\operatorname{argmin}} L_{\beta_0}(\mu, V)$$

The measurement $D_{\beta_0}(\beta)$ is of the generalization performance of an estimatorat $\beta = \beta_0$.

To serve our purpose, the K-fold cross validation (CV) method can be used which is simple and popular among practitioners (see Hastie, Tibshirani and Friedman, 2001). In K-fold CV method, one part of the available data is used for the estimation and a different part for evaluation. For the current problem, the K-fold CV method can be employed as a generalization scheme. It needs to split the data into K approximately equal-sized and homogeneous sections. After estimation from $K-1$ parts of the data, the β-divergence for the $K-1$th section is calculated. The calculated values of β_0-divergence is then combined to obtain the CV estimate.

The procedure to find the K-fold CV estimate $\widehat{D}_{\beta_0}(\beta)$ is summarized below.

- Split the data set into k subsets:

  $\{\wp(1), \dots, \wp(K)\}$.

  Let $\wp^{-k} = \{x_t | x_t \notin \wp(K)\}$, for $k = 1, \dots, K$

- Estimate μ and V using dataset $\wp^{-k}$ by

  i. Minimizing $D_\beta\left(\wp(x_t), \varphi_{u,v}(x_t)\right)$

$\Longrightarrow (\hat{\mu}_\beta, \hat{V}_\beta) = arg\min_{\mu,V} L_\beta(\ \mu, V)$

- Compute $CV_{(k)}$ using data set $\wp(k)$,

  ii. $CV_{(k)} = L_{\beta_0}(\hat{\mu}_\beta, \hat{V}_\beta)$.

  iii. Then $\widehat{\beta} = arg\min_{\beta} \widehat{D}_{\beta_0}(\beta)$

  iv. where $\widehat{D}_{\beta_0}(\beta) = \frac{1}{n}\sum_{k=1}^{n} CV_{(k)}$

- Compute $SD_{\beta_0}(\beta) = \mathrm{SE}\frac{1}{|\wp(\mathrm{k})|}\mathrm{C}V_{(k\,)}$ as a measure for variation of $\widehat{D}_{\beta_0}(\beta)$ where $|\wp(\mathrm{k})|$ denotes the number of elements in the k_{th} part of data $\wp(\mathrm{k})$ .Plots of $\widehat{D}_{\beta_0}(\beta)$ for β with the auxiliary boundary curves $\widehat{D}_{\beta_0}(\beta) \pm SD_{\beta_0}(\beta)$ will help us to select an optimum.

Robust Kernel Density Estimator (RKDE)

The standard kernel density estimator (KDE) for the sample X_t is given by

$$\hat{f}_{KDE}(x) = \frac{1}{n}\sum_{t=1}^{n} K_H(x, X_t) \tag{5.20}$$

where K_H is a kernel function with bandwidth H. Using the kernel trick, the kernel function can be expressed as an inner product in the Hilbert space $\mathcal{H}$, such that

$$K_H(x, X_t) = \langle \Phi(x), \Phi(X_t) \rangle \tag{5.21}$$

Where Φ is the mapping function $\Phi: \mathbb{R}^d \to \mathcal{H}$ and $\langle . \rangle$ denotes the inner product. These yields

$$\hat{f}_{KDE}(x) = \frac{1}{n}\sum_{t=1}^{n} \langle \Phi(x), \Phi(X_t) \rangle \tag{5.22}$$

$$= \langle \Phi(x), \frac{1}{n}\sum_{\mathrm{t}=1}^{\mathrm{n}} \Phi(X_t) \rangle$$

$$= \langle \Phi(x), \hat{\mu}_{\Phi,ML} \rangle$$

which is the inner product between $\Phi(x)$ and the sample mean of $\Phi(X_t)$. Replacing the sample mean by the robust β-divergent estimate

$$\hat{\mu}_{\Phi} = \frac{\sum_{t=1}^{n} \psi_{\beta}(x,\mu,V)\Phi(x_t)}{\sum_{t=1}^{n} \psi_{\beta}(x,\mu,V)} \tag{5.23}$$

leads to robust KDE (RKDE)

$$\hat{f}_{KDE}(x) = \langle \Phi(x), \hat{\mu}_{\Phi} \rangle = \langle \Phi(x), \sum_{t=1}^{n} w_t \Phi(x_t) \rangle = \sum_{t=1}^{n} w_t K_H(x, X_t) \tag{5.24}$$

They can be obtained using iteratively re-weighted least squares (IRWLS) (Marrona et al., 2006) as

$$\hat{\mu}_{\Phi_{j+1}} = \frac{\sum_{n}^{t=1} \psi_{\beta}(x_t|\mu_j,V_j)\Phi(x_t)}{\sum_{t=1}^{n} \psi_{\beta}(x_t|\mu_j,V_j)} = \frac{\sum_{n}^{t=1} \psi_{\beta}(x_t|MD_j)\Phi(x_t)}{\sum_{t=1}^{n} \psi_{\beta}(x_t|MD_j)} \tag{5.25}$$

Algorithm

Step 1: Obtain robust estimators of μ and V and then estimate MD robustly

Step 2: Obtain an estimate of β and calculate $\hat{\mu}, \hat{V}, \hat{\mu}_{\Phi}$ defined in (5.14), (5.15) and (5.23) using the initial weight $w(0) = \frac{1}{2}MD^2$ and $\psi_{\beta}(x, \mu, V)$ defined in (5.16).

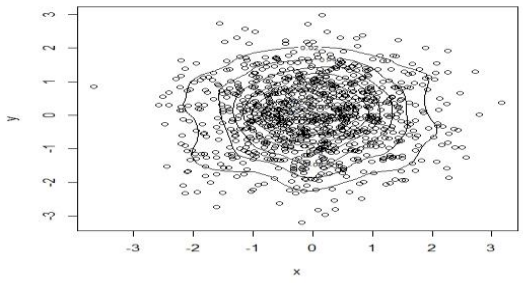
Step 3: Use an iterative reweighted least square to update the triple $\hat{\mu}, \hat{V}, \hat{\mu}_{\Phi}$ using equation (5.18), (5.19) and (5.25).

Stop when $|\hat{\mu}_{\Phi}(j+1) - \hat{\mu}_{\Phi}(j)| < \epsilon$.

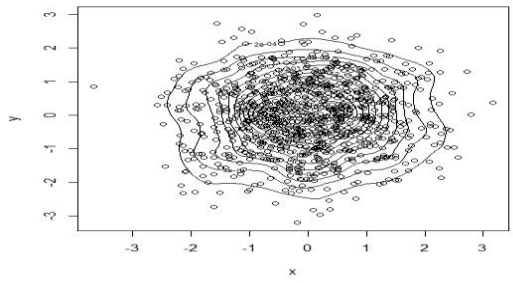
5.7 Monte-Carlo Simulation

To investigate the performance of the proposed RKDE in a comparison of the traditional KDE, we conduct a simulation study. We first investigate the effect of our adaptation for the outliyingness on kernel density estimation. We simulate 1000 observations from bivariate normal distribution with mean vector 0 and covariance identity. 5% of the simulated data are randomly replaced by normal variates with mean

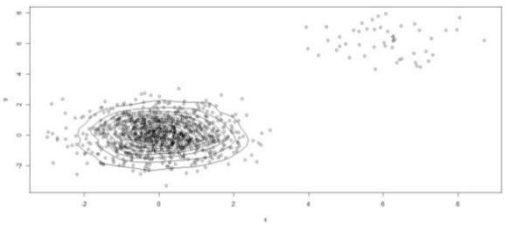
6 and same variance. We apply KDE and RKDE methods on these contaminated data. The process is repeated for 10%, 15% and 20% contamination. Figure 5.2 display the contour plots of KDE and RKDE for different level of contamination. We observe that both KDE and RKDE are not affected by 5% contaminated data. However, if the contamination level increases to 10%, 15% and 20% KDE provides bimodal distributions. On the other hand the RKDE is not influenced by the contamination even up to level 20%.



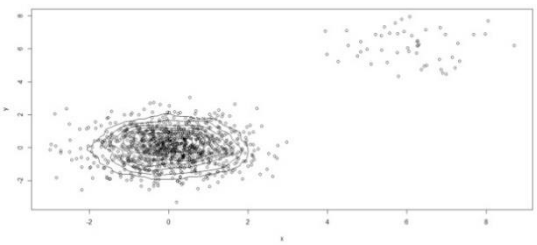
(a) KDE (No contamination)



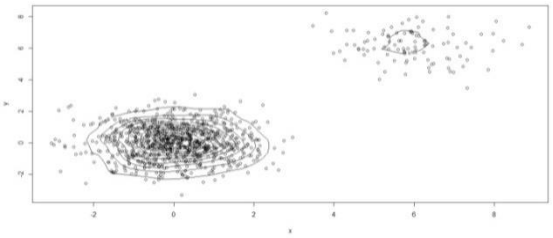
(b) RKDE (No contamination)



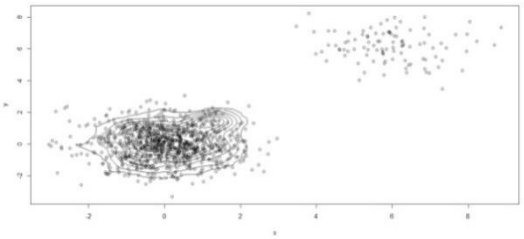
(c) KDE (5% contamination)



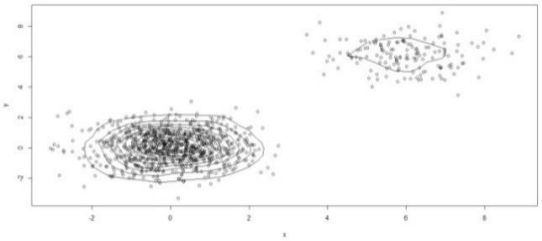
(d) RKDE (5% contamination)



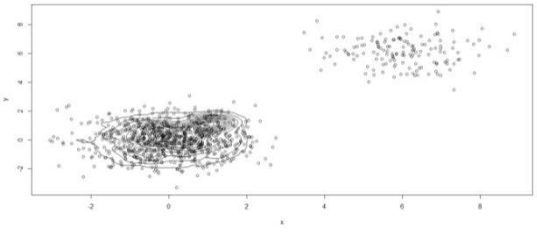
(e) KDE (10% contamination)



(f) RKDE (10% contamination)



(g) KDE (15% contamination)



(h) RKDE (15% contamination)

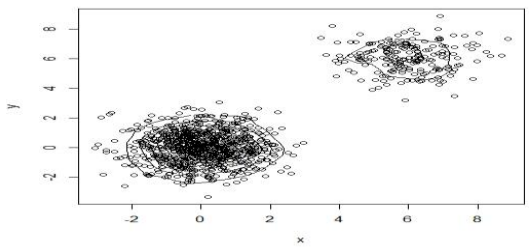
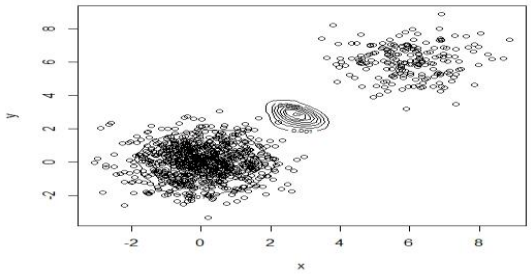
	
(i) KDE (20% contamination)	(j) RKDE (20% contamination)

Figure 5.2: Effect of outlier on kernel density

To demonstrate the validity of proposed algorithm we further simulate from bivariate normal and compute the precision (in terms of MSE) and the robustness (in terms of bias) in entropy estimation using KDE and RKDE for different level of contamination. The results are shown in Figure 5.3. We first compute the joint entropy for the clean data (no contamination) with KDE. We take this value of entropy as a standard for making comparison. For 5% contamination, the MSE and Bias of KDE, obtain from 1000 replications, is slightly higher than that of RKDE. However, as the contamination level increases to 10% or more, the MSE and Bias of KDE increases sharply. On the other hand, the MSE and Bias of RKDE increases gradually with the contamination level, but those are much lower than KDE for higher level of contamination.

We can conclude that the traditional kernel density estimator is robust up to certain level of contamination while our proposed robust kernel density estimator is highly robust and provide reliable output even for 20% contamination.

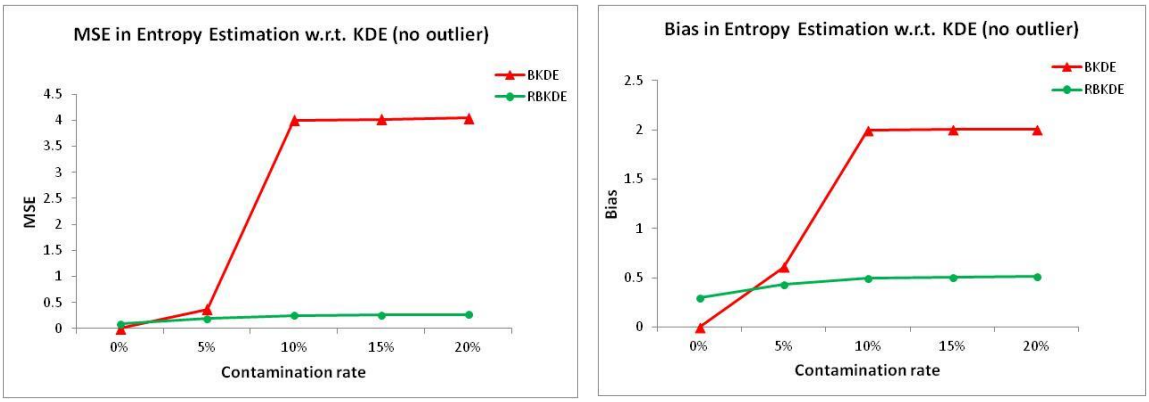


Figure 5.3: MSE and Bias in entropy estimation using kernel density

5.8 Application in portfolio

We provide an illustration of application of our proposed robust method in portfolio optimization. Recall the mean entropy entropy (MEE) model where the entropy is used as a risk measure of the asset and portfolio as well. This model needs to estimate the conditional entropies of assets for a given market index. In chapter 4, we use kernel density estimator to obtain the conditional entropies. Here, through an example, we will show that due to presence of outlier both the return and risk could result in bias estimate. The biasness is greatly reduced when we use the robust kernel density estimator for computing. For the illustration, we assume a portfolio based on four assets which are correlated to a market index. We, thus, simulate five variables from a multivariate skew normal distribution with mean, $\xi = [0\ 0\ 0\ 0\ 0]$, Covariance,

$$\Omega = \begin{bmatrix} 1 & 0 & 0 & 0 & .4 \\ 0 & 1 & 0 & 0 & .6 \\ 0 & 0 & 1 & 0 & .5 \\ 0 & 0 & 0 & 1 & .4 \\ .4 & .6 & .5 & .4 & 1 \end{bmatrix}, \text{ shape, } \alpha = [0.5\ 0.4\ 0.6\ 0.7\ 0.5].$$

We perform MEE portfolio analysis on the simulated data using both traditional kernel density estimator (KDE) and our robust kernel density estimator (RKDE). When the data is not contaminated, we found that with few exceptions, the RKDE provide estimates of mean, conditional entropy, portfolio weight, portfolio return and portfolio entropy quite close to those obtained by using KDE. We then contaminate the data with 5%. We then randomly replace 5% of the simulated data from multivariate skew normal distribution with same parameters except $\xi = [6\ 6\ 6\ 6\ 6]$. We observe that results of MEE for this contaminated data obtained using KDE vary a large from the results of clean data (Table 5.1). Specifically, the portfolio return and portfolio risk (entropy) are estimated as 0.66204 and 0.51532 which deviates from those estimated for clean data (0.37010 and 0.67669). Note that the portfolio return is over estimated while the

portfolio risk is underestimated. On the other hand when RKDE is used on contaminated data, the results are quite similar to that obtained from the clean data; for instance, the portfolio return and entropy are estimated as 0.32077 and 0.68043. The results suggest that if data are contaminated, use of KDE for entropy estimation in portfolio analysis may produce misleading results. Our proposed RKDE is a useful remedy in such cases.

Table 5.1: MEE Portfolio with KDE and RKDE

		Clean data		5% contaminated data	
		KDE	**RKDE**	**KDE**	**RKDE**
Mean	Asset 1	0.48395	0.46317	0.75297	0.44251
	Asset 2	0.19700	0.19015	0.50157	0.16721
	Asset 3	0.38634	0.36836	0.71447	0.34358
	Asset 4	0.42068	0.39143	0.68781	0.33572
Conditional Entropy	Asset 1	2.75516	2.77490	2.13381	2.77193
	Asset 2	2.62672	2.64319	1.98024	2.65096
	Asset 3	2.68927	2.70895	2.03760	2.70427
	Asset 4	2.76023	2.77856	2.10015	2.76327
Portfolio Weight	Asset 1	0.24591	0.24583	0.24224	0.24578
	Asset 2	0.25709	0.25723	0.25932	0.25621
	Asset 3	0.25151	0.2541	0.25265	0.25151
	Asset 4	0.24549	0.24553	0.24579	0.24650
Portfolio Return		0.37010	0.35149	0.66204	0.32077
Portfolio Entropy		0.67669	0.68132	0.51532	0.68043

We further investigate why the portfolio entropy is under estimated for contaminated data. Figure 5.4 displays the KDE of simulated data from N(0,1) distribution. The upper panel of the figure shows the KDE of clean data and the lower panel shows the KDE of the same distribution with one outlier. As we observe, the estimated density for contaminated data has large tail area with probability close to zero at more grid points than the estimated density for clean data has. Subsequently, an entropy estimate with contaminated data will be inflated since probability estimates are small at more grid points than it should be. This result confirms the rationality of using truncation for entropy estimation.

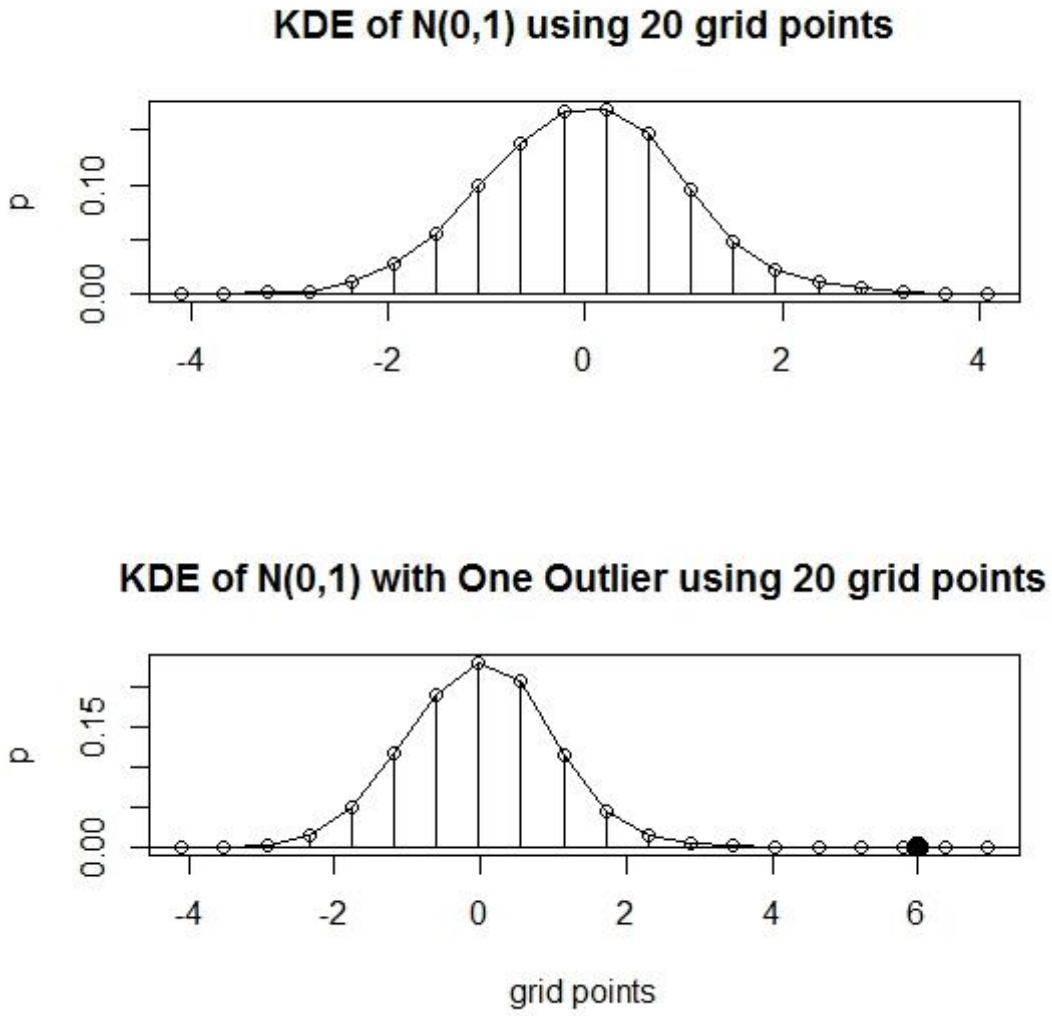


Figure 5.4: Effect of outlier on kernel density

CHAPTER 6: CONCLUSION

6.1 Summary of contribution

This study addressed two core problems of asset allocation: risk measure and portfolio diversification. Review on early literature reveals that traditional risk measures are subject to estimation error that results unstable portfolio allocation in practice. Diversity is, thus, essential to control the instability of asset allocation. We advocate for entropy, a nonparametric alternative of variance as a risk measure. We found that entropy can balance diversification to reach a good performance with reasonable risk. Relation between entropy and variance is studied for a range of probability distributions. It reveals that entropy is equivalent to variance for most of the return distributions, however, the benefit of entropy is that it not restricted to the assumption of normality. Investigation on real data suggests that asymmetry and heaviness of tail are common in return distributions. Entropy, therefore, is in advantageous position since it depends on many more parameter than variance and contains more information regarding data distribution. We further argue that like variance entropy measure can capture the effect of diversity as a risk measure. However, estimation of entropy is not an easy task, especially, when the data distribution is unknown. This study discussed different steps of entropy estimation from real and simulated data. We note that the accuracy of entropy estimation depends on density estimation. Different methods of density estimation are compared; the technical details like bandwidth selection for kernel density and bin selection for histogram are provided in this thesis with computer codes.

We proposed a class of portfolio policies that have better stability properties than the traditional minimum-variance portfolio. The portfolio weights of the resulting policies are less sensitive to changes in the distributional assumptions than those of the traditional minimum-variance policy. Our proposed an entropy-based new multi-

objective portfolio model is compared with existing models using a rolling window procedure. Our numerical results on simulated data reveal that the proposed model is more stable (diversified) and that they preserve (or slightly improve) the already relatively high out-of-sample Sharpe ratio of the minimum-variance policy when data comes from normal distribution. The proposed model is further evaluated using equity market data. A variety of performance measures confirm that this model is more diversified than its competitors and provide better performance in both in-and out-of-sample cases.

We then investigated the robustness of entropy measure and found that like variance entropy is sensitive to outlier. We proposed a robust procedure of entropy estimation based on kernel density estimator (KDE). The β-divergent principal is first utilized to obtain a robust kernel density estimator (RKDE). The RKDE is a weighted kernel density estimate, where smaller weights are given to more outlying data points. Our simulation results suggest that our proposed method has lower root mean square error than the classical KDE. Entropy from RKDE is then estimated with a truncation procedure to render the effect of outlier. A numerical example is given to illustrate the idea of our model and demonstrate the effectiveness of the designed algorithm. The computational results show that the proposed model and the designed algorithm are reliable and provide greater accuracy in estimating portfolio weights.

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A Nonparametric and Diversified Portfolio Model

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Abstract. Traditional portfolio models, like mean-variance (MV) suffer from estimation error and lack of diversity. Alternatives, like mean-entropy (ME) or mean-variance-entropy (MVE) portfolio models focus independently on the issue of either a proper risk measure or the diversity. In this paper, we propose an asset allocation model that compromise between risk of historical data and future uncertainty. In the new model, entropy is presented as a nonparametric risk measure as well as an index of diversity. Our empirical evaluation with a variety of performance measures shows that this model has better out-of-sample performances and lower portfolio turnover than its competitors.

Keywords: Portfolio optimization, Entropy, Risk measure, Diversified portfolio.
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INTRODUCTION

Risk and diversification are two fundamental concepts in asset allocation. Modern portfolio theory begins with mean-variance (MV) model of Markowitz [1], which is well accepted among investment communities and has enjoyed its reputation until today. The instability and ambiguity of MV optimization is that it magnifies the impact of estimation errors [2]. The success of the portfolio thus partially depends on the proper estimate of the risk.

Though risk may be well estimated from historical data, the problem that MV portfolio often concentrates only on few assets may not be resolved. Therefore, an optimal portfolio under MV criteria may not offer better out-of-sample performance than the naive 1/N benchmark [3]. Moreover, MV is restricted to the normally distributed assets, which are characterized by the first two moments only.

Entropy and information theory analysis has received numerous attentions from researchers in Finance and Economics. Entropy as a nonparametric measure of portfolio risk is first introduced by Philippatos and Wilson [4] and Philippatos and Gressis [5] with a conclusion that if the asset distributions are either normal or uniform, the mean-entropy and mean-variance portfolios are equivalent. Entropy is not only nonparametric but also more informative as it depends on higher order moments than variance [6]. Dionisio et al. [7] further demonstrate that similar to variance, entropy of a portfolio decreases as the number of assets increases. Thus, in capturing meaningful information from the system, entropy has some distinguish features than ordinary statistical tools like variance. In a multi-objective portfolio model, Samanta and Roy [8], followed by Jana et al. [9, 10] and Ke and Zhang [1], utilize entropy to obtain a well-diversified portfolio.

In this paper, we propose a portfolio model, which is well-diversified as well as based on a nonparametric risk measure. In the proposed model, entropy is presented as a measure of risk and also as an index of diversity. The rest of the paper is organized as follows. In next section, with a brief introduction of entropy, we demonstrate the evolution of MV portfolio model in relation with entropy. A rational extension of the mean-entropy portfolio is proposed. We then evaluate the performance of the new model and compared with some benchmark models. The paper ends with some concluding remarks.

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Entropy-based portfolio models: Practical issues

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ABSTRACT

Entropy is a nonparametric alternative of variance and has been used as a measure of risk in portfolio analysis. In this paper, the computation of entropy risk for a given set of data is discussed with illustration. A comparison between entropy-based portfolio models is made. We propose a natural extension of the mean entropy portfolio to make it more general and diversified. In terms of performance, this new model is similar to the mean-entropy portfolio when applied to real and simulated data, and offers higher return if no constraint is set for the desired return; also it is found to be the most diversified portfolio model.

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APPENDIX A

ENTROPY ESTIMATION FROM GIVEN DENSITY

Now we discuss how we can compute univariate, joint and conditional entropy with R when the density is given. The following R function can be used to calculate univariate Shannon entropy

```
# this function compute entropy from a univariate density p

uni_entp= function(p,c=1){

Err=1e-6;

if (abs(1-sum(p))>Err) print("Input vector must be a probability mass function")

# replace 0s in p by 1 to avoid log(0)=inf

h1=-rep(1,length(p))

h2=((as.logical(p)+h1)*h1)+p

H=h2%*%(log(h2))*(-c)

return(H)}
```

The above function is an implementation of integral plug-in estimate of entropy. The resubstitution estimate can be obtained by simple modification of the above function. For a resubstitution estimate, we just need to replace

```
H=h2%*%(log(h2))*(-c)        by

H=sum(1/(length(p)))*(log(h2))*(-c)
```

To make use of splitting data estimate we need to divide sample observations {X} into two subsamples say $\{X_l\}$ and $\{X_m\}$. $\{X_l\}$is used to estimate the density and $\{X_m\}$is used to estimate entropy. Below is an example:

```
#simulate from normal

n=20

X=rnorm(n,0,1)

#split the data
```

```
l=floor(n/2)
m=n-l
Xl=X[1:l]
Xm=X[l+1:m]
    #use first subsample to estimate a kernel density
    library(ks)
    bw=hpi(Xl, nstage=2)
    dnc=kde(x=Xl, h=bw)
    p=dnc$estimate
    p=p/sum(p)
    A=dnc$eval.points
    #function to calculate entropy from previously estimated density
    entp.split=function(A,p,x){
    d=matrix(1,length(x),length(A))
    xx=x*d
    AA=matrix(rep(A,length(x)),nrow=length(x),byrow = TRUE)
    span=A[2]-A[1]
    ind=(AA-xx<span)&(AA-xx>=0)
    pp=ind*p
    h1=-matrix(1,dim(pp)[1],dim(pp)[2])
    h2=((as.logical(pp)+h1)*h1)+pp
    H=-(1/length(x))*sum(log(h2))
    return(H)
    }
    #use the function to estimate entropy
    source("…\\entp.split.r")
```

```
H=entp.split(A=A ,p=p, x=Xm)
```

Now, we state how to compute univariate entropy using cross validation plug-in estimate. Suppose we have a set of observations:

```
X=rnorm(10,0,1)
```

First, we need to estimate the probabilities from the observations removing one observation at a time and identify the probability for each observation. Then use the formula given in (8). Below are the R codes:

```
#this function calculate probablity for each observation
#by cross validation method
#X is the set of all observations without x
cross.p=function(X,x){
  library(ks)
bw=hpi(X, nstage=2)
dnc=kde(x=X, h=bw)
p=dnc$estimate
A=dnc$eval.points
d=matrix(1,length(x),length(A))
xx=x*d
AA=matrix(rep(A,length(x)),nrow=length(x),byrow = TRUE)
span=A[2]-A[1]
ind=(AA-xx<span)&(AA-xx>=0)
pp=ind*p
pcv=sum(pp)
return(pcv)
}
#this function calculate univariate entropy by cross validation plug-in estimate
```

```
entp.crossv=function(X){
n=length(X)
library(base)
library(ks)
source("Y:\\functionY\\cross.p.r")
XX=matrix(0, n, n-1)
pcv=rep(0,n)
for (i in 1:n){
XX[i,]=setdiff(X,X[i])
pcv[i]=cross.p(X=XX[i,],x=X[i])
}
H=-(1/n)*sum(log(pcv))
return(H)
}
```

The plug-in estimate of joint entropy is a straightforward extension of univariate entropy. Following is an R function for computing joint entropy from the joint density by integral plug-in estimate.

```
# this function compute joint entropy using integral plug-in estimate
#from the joint density p
entr.join=function(p,c=1){
# m is the number of states in y
m=dim(p)[1]
# n is the number of states in x
n=dim(p)[2]
# verify that p is a density function
Err=1e-6
```

```
if (abs(1-sum(sum(p)))>Err) print("Input matrix must be a joint distribution")
# replace 0s in p by 1 to avoid log(0)=inf
h1=-matrix(1,m,n)
h2=((as.logical(p)+h1)*h1)+p
# compute p log(p)
h3=p*log(h2)
H=-c*sum(sum(h3))
return(H)
}
```

Now, we will describe how to compute conditional entropy when the joint density, *p* of X and Y is given. Suppose the joint density *p* is a matrix whose columns represent levels of X and rows represent level of Y. Below are R codes for computing conditional entropy H(X|Y).

```
entr.cond=function(p){
p1=colSums(p) # marginal density of X
p2=rowSums(p) # marginal density of Y
m=length(p2) # number of rows
l=length(p1) #number of columns
pc1=matrix(0,m,l)
Hc1=matrix(0,m,1)
for (i in 1:m) {
pc1[i,] = p[i,]/sum(p[i,]) #conditional density of X given y(i)
source("Y:\\functionY\\uni_entp.r")
Hc1[i]=uni_entp(pc1[i,]) #conditional Entropy of X given y(i)
}
H1=sum(p2*Hc1) # conditional Entropy of X given Y
return(H1)       }
```

APPENDIX B

PROOF OF SUBADDIVITIVITY OF ENTROPY

$$\boldsymbol{H(X,Y) \leq H(X) + H(Y)}$$

$$H(X,Y) - H(X) - H(Y)$$

$$= \sum_{x,y} p(x,y) \log(\frac{1}{p(x,y)}) - \sum_{x} p(x) \log(\frac{1}{p(x)}) - \sum_{y} p(y) \log(\frac{1}{p(y)}$$

$$= \sum_{x,y} p(x,y) \log(\frac{1}{p(x,y)}) - \sum_{x,y} p(x,y) \log(\frac{1}{p(x)}) - \sum_{x,y} p(x,y) \log(\frac{1}{p(y)})$$

$$= \sum_{x,y} p(x,y) \log(\frac{p(x)p(y)}{p(x,y)})$$

$$\leq \log\left(\sum_{x,y} \frac{p(x,y)p(x)p(y)}{p(x,y)}\right) \sum_{x,y} p(x)p(y) = 1$$

$$= \log 1 = 0$$

Where the inequality follows from Jensen's inequality applied to the convex function $\log\left(\frac{1}{x}\right)$□